M 332 Unit 16

THE OPEN UNIVERSITY
Complex Analysis
Mathematics : A Third Level Course

Unit 16 Boundary Value Problems

COMPLEX ANALYSIS

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

Complex Analysis Unit 16

Boundary Value Problems

Prepared by the Course Team

The Open University Press

The Open University Press, Walton Hall, Milton Keynes.

First published 1975

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Produced in Great Britain by
Technical Filmsetters Europe Limited, 76 Great Bridgewater Street, Manchester M1 5JY

ISBN 0 335 05557 5

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1.1

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Conventions

Before working through this text make sure you have read *A Guide to the Course: Complex Analysis.*

References to units of other Open University courses in mathematics take the form:

Unit M100 13, Integration II.

The set book for the course M231, Analysis, is M. Spivak, *Calculus*, paperback edition (W. A. Benjamin/Addison-Wesley, 1973); this is referred to as

Spivak.

Optional Material

This course has been designed so that it is possible to make minor changes to the content in the light of experience. You should therefore consult the supplementary material to discover which sections of this text are not part of the course in the current academic year.

Reminder: This unit is one of the last four units in the course. You are not required to study all of these four units, so you should consult the supplementary material to see what choice is available to you.

16.0 INTRODUCTION

The aim of this unit is to teach you some techniques for solving a certain class of problems in applied mathematics, namely: boundary value problems for Laplace's equation in two dimensions, which is $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$. Because this course has not so far been concerned with applications of mathematics, the approach and attitude of this unit may be a little unfamiliar to you, if you have not studied, for example, MST282, Mechanics and Applied Calculus, or M201 Linear Mathematics, or M321, Partial Differential Equations of Applied Mathematics. However, you do not require any *knowledge* from such courses in order to understand this unit. Nevertheless, if you want to enjoy it, then you should be sympathetic to the idea of applying mathematical techniques to solve practical problems.

To get an idea of the flavour of this unit, glance through Example 5 and its solution on pages 49–50 of Section 16.7. (This example is concerned with the flow of a fluid over a very large plane at the centre of which is a circular obstacle. We are told that the velocity of the fluid far from the obstacle is constant, and we require to find the velocity of the fluid at all points (by finding G) and to give a picture which represents the flow of the fluid (by sketching streamlines).) This is typical of the sort of problem that we want to solve. In order to get to the point of doing so, we have to spend some time defining various concepts and proving various theorems. For example, the words "fluid flows" are used in the statement of the example: in Section 16.6 we set up a mathematical model of fluid flows. We also mention "velocity" in the example: the representation of velocity and other *vectors* is dealt with in Section 16.4, which also explains what "potential" is and discusses conditions under which it exists. Underlying much of the work, and appearing through symbols such as $\frac{\partial \phi}{\partial n}$, is a stratum of two-variable calculus: we give you the necessary tools in Section 16.1. Most importantly, the phrase "analytic equivalence" and the use of complex functions show that the whole approach relies on complex analysis: relevant results from complex analysis and their translation into terms of two-variable calculus form the material of Section 16.2.

There are a lot of new terms in this unit: we do not expect you to commit them all to memory. We have prepared a partial glossary of terms, which you will find in the accompanying supplementary material.

The major part of this unit, so far as importance, and assessment, is concerned, comprises Sections 16.7 and 16.8, and you should aim to spend about one third of your time on this. Next in importance comes Sections 16.2 and 16.3. The remaining sections introduce a variety of less important concepts and techniques that you will need in order to study the important material: much of it you should find fairly elementary, so try to work through it fairly quickly.

One final word about the problems. These are basically of two kinds: the first develops and reinforces aspects of the theory, while the second deals with practical techniques. The weight of continuous and examination assessment will bear mainly on problems of this second kind. You could regard Problems 1 and 4 of Section 16.3, Problem 1 of Section 16.5, and all the problems of Section 16.8 as being typical of their kind.

Objectives

After working through this unit you should be able to:

(i) use and understand the terms listed in the partial glossary (see the supplementary material);

(ii) determine whether a given function is harmonic on a given region;

(iii) find harmonic conjugates of given harmonic functions;

(iv) determine grad ϕ, div F and curl F, for given functions ϕ and F;

(v) solve given Dirichlet, Neumann and Dirichlet–Neumann problems by direct means, by the method of images, and by the use of the table of analytic equivalences in Section 16.9.

Results Needed from Earlier Units

Unit 3: The Cauchy–Riemann Equations; in particular, Theorem 15.
Unit 5: Problem 6 of Section 5.8.

Unit 9: The Local Maximum Principle (Theorem 1).
The Boundary Uniqueness Theorem (Theorem 2).
The Antiderivative Theorem (Theorem 4).
Cauchy's Theorem (Theorem 5).
Cauchy's Formula (Theorem 7).
Problem 3 of Section 9.2.
Problem 5 of Section 9.2.

Television

There is a television programme associated with this unit.

16.1 PRELIMINARIES ON TWO-VARIABLE CALCULUS

In this course we have assumed no knowledge of what is usually called two-variable calculus, except for a little knowledge of partial derivatives required in Section 3.8 of *Unit 3, Differentiation.* In this unit we shall require a little more knowledge from time to time, and it is the purpose of this section to provide this. Further details, for those who are interested, can be found in the Appendices to *Units 2, 3, 4, 5* and *7* of M231 Analysis.

Recall that a function F is called *real-valued* if the range of F is contained in $\mathbf{R}$. We shall consider mainly real-valued functions in this section. It is possible to draw the graph of a real-valued function F with domain a subset of $\mathbf{C}$ by sketching the surface $\{(x, y, u): u = F(x + iy)\}$ in three-dimensional space $\mathbf{R}^3$. However, a more useful representation for our purposes is in terms of level curves. Given a real-valued function F, and a real number k, we define the **level curve of** F **at height** k to be the set $\{(x, y): F(x, y) = k\}$. (Note that, as remarked in *Unit 2, Continuous Functions,* we use (x, y) or $x + iy$ as a typical element of the domain of F, according to taste.) The level curve need not be a curve in the accepted sense; however, if F is "reasonable", then the level curves will be *contours.* We discuss this point again at the end of this section.

Recall that we can define the *partial derivatives* of the real-valued function F *at* the point (a, b), denoted by $\dfrac{\partial F}{\partial x}(a, b)$ and $\dfrac{\partial F}{\partial y}(a, b)$, to be the real numbers

$$\frac{\partial F}{\partial x}(a, b) = \lim_{h \to 0} \frac{F(a + h, b) - F(a, b)}{h},$$

and

$$\frac{\partial F}{\partial y}(a, b) = \lim_{k \to 0} \frac{F(a, b + k) - F(a, b)}{k} \qquad (\text{where } h, k \in \mathbf{R}).$$

The *functions* $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ are called the *partial derivatives* of F. Note that in terms of complex numbers, if $\alpha = (a, b)$, then

$$\frac{\partial F}{\partial x}(\alpha) = \lim_{h \to 0} \frac{F(\alpha + h) - F(\alpha)}{h} \quad \text{and} \quad \frac{\partial F}{\partial y}(\alpha) = \lim_{k \to 0} \frac{F(\alpha + ik) - F(\alpha)}{k}.$$

One point about notation is worth making: the x in $\dfrac{\partial F}{\partial x}$ is not a particular number, but has the same meaning as the x in "x-axis"; essentially it stands for a particular *direction.*

The functions $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ are real-valued functions, and so may well have partial derivatives. We often print $\dfrac{\partial G}{\partial x}$ as $\dfrac{\partial}{\partial x}(G)$ if G is a complicated expression: thus the partial derivatives of $\dfrac{\partial F}{\partial x}$ are $\dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial x}\right)$ and $\dfrac{\partial}{\partial y}\left(\dfrac{\partial F}{\partial x}\right)$, which are abbreviated normally to $\dfrac{\partial^2 F}{\partial x^2}$ and $\dfrac{\partial^2 F}{\partial y \partial x}$; similarly, the partial derivatives of $\dfrac{\partial F}{\partial y}$ are $\dfrac{\partial^2 F}{\partial x \partial y}$ and $\dfrac{\partial^2 F}{\partial y^2}$. These four partial derivatives of $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ are called the *second partial derivatives* of F, and will play a very important role in this unit.

Example 1

Let $F(x, y) = x^2 + xy + y^3$. Calculate the first and second partial derivatives of F.

Solution

The partial derivatives are given by

$$\frac{\partial F}{\partial x}(x, y) = 2x + y, \quad \frac{\partial F}{\partial y}(x, y) = x + 3y^2;$$

the second partial derivatives are given by

$$\frac{\partial^2 F}{\partial x^2}(x, y) = 2, \quad \frac{\partial^2 F}{\partial y \partial x}(x, y) = 1, \quad \frac{\partial^2 F}{\partial x \partial y}(x, y) = 1, \quad \text{and} \quad \frac{\partial^2 F}{\partial y^2}(x, y) = 6y.$$

It is not an accident that $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$, as you will see shortly.

There are six important properties of partial derivatives that we shall need. These are:

(1) the Constancy Condition,

(2) the Chain Rule,

(3) the Commuting Theorem on the order of partial differentiation,

(4) the Analyticity Criterion,

(5) the Directional Derivative Formula,

and (6) the Implicit Function Theorem on level curves.

We shall discuss these in turn on the next few pages. Since the various results we discuss are peripheral, though necessary to our main purpose, we have given them descriptive names rather than theorem numbers, and have omitted the proofs.

(1) It is a well-known result of real analysis that if a real function f has zero derivative on an open interval I, then f is constant on I (it is an easy consequence of the Mean Value Theorem). A similar result holds for two variables.

The Constancy Condition

Suppose that F is a real-valued function continuous on a region S, and the partial derivatives of F exist and satisfy

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \text{ on } S.$$

Then F is constant on S.

(2) Let $F:S \longrightarrow \mathbf{R}$ be a real-valued function having partial derivatives on the region S. We can compose F with a differentiable function $h:\mathbf{R} \longrightarrow \mathbf{R}$ to obtain $h \circ F:S \longrightarrow \mathbf{R}$. The partial derivatives of $h \circ F$ can be calculated using the Chain Rule (**Spivak**, page 154). For let $(a, b) \in S$, and $G = h \circ F$; and let us consider $\frac{\partial G}{\partial x}(a, b)$. Now it is easy to check that $\frac{\partial G}{\partial x}(a, b) = g'(a)$, where $g(x) = G(x, b)$. Also $g(x) = h(F(x, b)) = h(f(x))$, where $f(x) = F(x, b)$, and so $g'(a) = h'(f(a))f'(a)$ by the Chain Rule; so that

$$\frac{\partial G}{\partial x}(a, b) = h'(F(a, b)) \cdot \frac{\partial F}{\partial x}(a, b).$$

In other words, we have

$$\frac{\partial}{\partial x}(h \circ F) = (h' \circ F) \cdot \frac{\partial F}{\partial x},$$

and, of course, a similar equality involving $\frac{\partial}{\partial y}$.

However, there is *another* form of the Chain Rule that is important for us, which applies to smooth arcs. (Note that from here on, as usual, U and V denote real-valued functions, and occur as the real and imaginary parts, respectively, of complex functions.)

Chain Rule (for smooth arcs)

Let S be a region, $[c, d]$ a closed interval and $\phi: [c, d] \longrightarrow S$ be a smooth arc with $\phi = U + iV$, and $F: S \longrightarrow \mathbf{R}$ have continuous partial derivatives on S. Then $F \circ \phi$ is differentiable on $[c, d]$ (in the sense of real analysis), and

$$(F \circ \phi)' = \left(\frac{\partial F}{\partial x} \circ \phi\right) U' + \left(\frac{\partial F}{\partial y} \circ \phi\right) V'.$$

Example 2

Let $\phi(t) = e^{it}$, and $F(x, y) = x^2 - y^2$. Calculate the derivative of $F \circ \phi$ by (a) the Chain Rule for smooth arcs, and (b) direct substitution. (Note that in this case $S = \mathbf{C}$ and $[c, d] = [0, 2\pi]$; we do not normally need to state explicitly what S and $[c, d]$ are.)

Solution

(a) $\dfrac{\partial F}{\partial x}(x, y) = 2x$ and $\dfrac{\partial F}{\partial y}(x, y) = -2y$: also $U(t) = \cos t$ and $V(t) = \sin t$, so that $U'(t) = -\sin t$ and $V'(t) = \cos t$. Thus

$$\begin{aligned}(F \circ \phi)'(t) &= (2 \cdot \cos t)(-\sin t) + (-2 \cdot \sin t) \cos t \\ &= -4 \sin t \cos t.\end{aligned}$$

(b) Now by direct substitution of $x = \cos t$, $y = \sin t$ into $F(x, y)$,

$$(F \circ \phi)(t) = \cos^2 t - \sin^2 t = \cos 2t,$$

and so $(F \circ \phi)'(t) = -2 \sin 2t = -4 \sin t \cos t$, as expected.

---

(3) There is a general result which tells us that we can interchange the order of partial differentiation under certain conditions.

The Commuting Theorem

Let F be a real-valued function which has second partial derivatives continuous on a neighbourhood of the point (a, b). Then

$$\frac{\partial^2 F}{\partial x \partial y}(a, b) = \frac{\partial^2 F}{\partial y \partial x}(a, b).$$

(4) Theorem 15 of *Unit 3* gave conditions under which a function was analytic. We restate it here (in a form suited to the present discussion).

The Analyticity Criterion (Theorem 15 of Unit 3)

If $f = U + iV$, where U and V are real-valued functions having partial derivatives defined on a neighbourhood of α and continuous at α, and the Cauchy–Riemann equations

$$\frac{\partial U}{\partial x}(\alpha) = \frac{\partial V}{\partial y}(\alpha), \quad \frac{\partial V}{\partial x}(\alpha) = -\frac{\partial U}{\partial y}(\alpha)$$

are satisfied, then f is differentiable at α (in the sense of complex analysis), and furthermore

$$f'(\alpha) = \frac{\partial U}{\partial x}(\alpha) + i\frac{\partial V}{\partial x}(\alpha) = \frac{\partial V}{\partial y}(\alpha) - i\frac{\partial U}{\partial y}(\alpha).$$

(5) The **directional derivative** of the real-valued function F **at** α **in the direction** θ, denoted by $\nabla_\theta F(\alpha)$ (where ∇ is pronounced "del"), is defined to be

$$\nabla_\theta F(\alpha) = \lim_{t\to 0} \frac{F(\alpha + te^{i\theta}) - F(\alpha)}{t}, \quad \text{where } t \in \mathbf{R}.$$

This measures the rate of increase of F at α in the direction θ (Fig. 1).

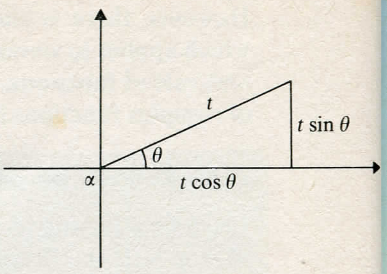


Fig. 1

In "two-variable" language, with $\alpha = (a, b)$,

$$\nabla_\theta F(\alpha) = \lim_{t\to 0} \frac{F(a + t\cos\theta, b + t\sin\theta) - F(a, b)}{t}.$$

Note that $\nabla_0 F = \dfrac{\partial F}{\partial x}$, and $\nabla_{\pi/2} F = \dfrac{\partial F}{\partial y}$, which is just as well. For a general θ, we can relate $\nabla_\theta F$ to $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ by using the Chain Rule. Let $\phi(t) = \alpha + te^{i\theta}$, $t \geqslant 0$, be a parametrization of the ray from α with direction θ. Then $\nabla_\theta F(\alpha) = (F \circ \phi)'(0)$. If $U =$ Re ϕ, $V =$ Im ϕ, then $U(t) = a + t\cos\theta$, $V(t) = b + t\sin\theta$, so that

$$\nabla_\theta F(\alpha) = \frac{\partial F}{\partial x}(\alpha)\cdot\cos\theta + \frac{\partial F}{\partial y}(\alpha)\cdot\sin\theta,$$

by the Chain Rule. This result is called the **Directional Derivative Formula.**

(6) The last result concerns level curves of a real-valued function. These form smooth arcs, under circumstances given by the following theorem. If you find the statement difficult at a first reading, read Example 3 next, and not the paragraph following the statement of the theorem.

The Implicit Function Theorem

Let S be a region and $F: S \longrightarrow \mathbf{R}$ be a continuous real-valued function, with partial derivatives continuous on S. Let $\alpha = a + ib \in S$, $k = F(\alpha)$, and let at least one of $\dfrac{\partial F}{\partial y}(\alpha)$ and $\dfrac{\partial F}{\partial x}(\alpha)$ be non-zero.

(i) If $\dfrac{\partial F}{\partial y}(\alpha) \neq 0$, there is an open rectangle R containing α and a real function f differentiable at a such that, for $(x, y) \in R$, $y = f(x) \Leftrightarrow F(x, y) = k$.

(ii) If $\dfrac{\partial F}{\partial x}(\alpha) \neq 0$, there is an open rectangle R containing α and a real function g differentiable at b such that, for $(x, y) \in R$, $x = g(y) \Leftrightarrow F(x, y) = k$.

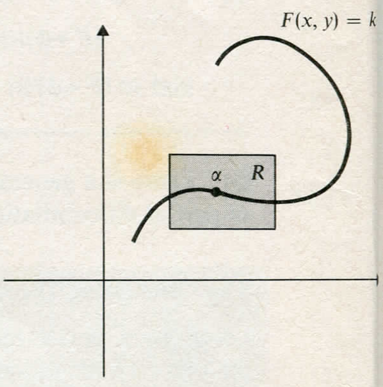


Fig. 2

Thus, there is a neighbourhood of α in which the level curve $F(x, y) = k$ is the path of a differentiable arc ϕ (in the sense of *Unit 4, Integration*). For if $\dfrac{\partial F}{\partial y}(\alpha) \neq 0$, we can define $\phi(t) = \big(t, f(t)\big)$, whereas if $\dfrac{\partial F}{\partial x}(\alpha) \neq 0$, then $\phi(t) = \big(g(t), t\big)$.

We can calculate the derivative of f by applying the Chain Rule to F and the function $\phi(x) = \big(x, f(x)\big)$. Since $(F \circ \phi)(x) = F\big(x, f(x)\big) = k$, $(F \circ \phi)'(a) = 0$, and so

$$\frac{\partial F}{\partial x}\big(\phi(a)\big)\cdot 1 + \frac{\partial F}{\partial y}\big(\phi(a)\big)\cdot f'(a) = 0,$$

since when $\phi = U + iV$, $U(x) = x$ and $V(x) = f(x)$.

Thus $\dfrac{\partial F}{\partial x}(\alpha) + \dfrac{\partial F}{\partial y}(\alpha) f'(a) = 0$, so that $f'(a) = -\dfrac{\partial F}{\partial x}(\alpha)\Big/\dfrac{\partial F}{\partial y}(\alpha)$. A similar result holds for g.

Example 3

Let $F(x, y) = x^2 + xy + y^3$, $S = \mathbf{C}$, $\alpha = (1, -2)$, $k = -9$. Verify whether the hypotheses of the Implicit Function Theorem hold for the above specifications.

Solution

The function F is continuous and real-valued on S, and has continuous partial derivatives on S since $\frac{\partial F}{\partial x}(x, y) = 2x + y$ and $\frac{\partial F}{\partial y}(x, y) = x + 3y^2$. Also

$$F(1, -2) = -9, \quad \frac{\partial F}{\partial x}(1, -2) = 0 \quad \text{and} \quad \frac{\partial F}{\partial y}(1, -2) = 13.$$

Hence, case (i) is satisfied; so there is a function f defined on a neighbourhood of 1 such that $f(x) = y \Leftrightarrow x^2 + xy + y^3 = -9$. Also f is differentiable at 1, with derivative $-0/13 = 0$. Note that there is no very nice explicit formula for f.

You should not worry too much about mastering the precise hypotheses of the various theorems in this section; all you will need later is a feeling for the kind of situation in which they might apply, and some practice in using any associated techniques. The Self-Assessment Questions are designed to cover these aims.

Summary

In this section we refreshed your memory of partial differentiation and stated six results on partial derivatives which we shall require later. These were the Constancy Condition, the Chain Rule, the Commuting Theorem, the Analyticity Criterion, the Directional Derivative Formula, and the Implicit Function Theorem.

Self-Assessment Questions

1. Sketch some of the level curves of the functions (a) $F(z) = \log|z|$ and (b) $G(z) = \operatorname{Arg} z$.
2. Define "partial derivative".
3. Let $F(x, y) = \sin(x^2 + y)$. Calculate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.
4. Use the Cauchy–Riemann equations to show that the function

   $$f(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

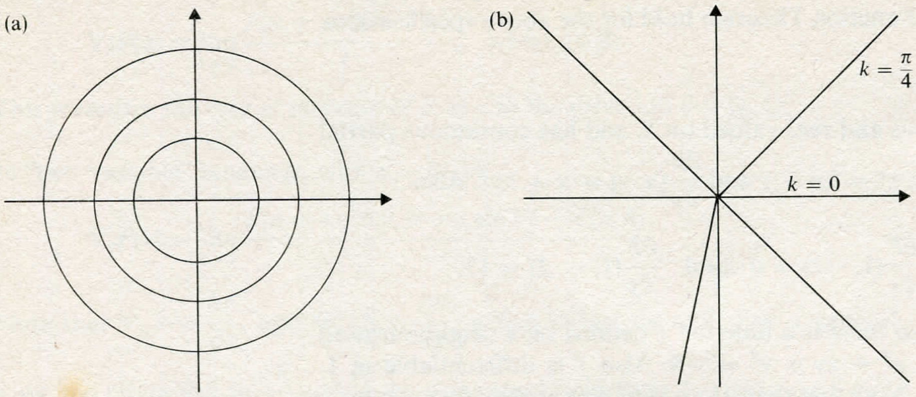
   is entire.
5. Let $F(x, y) = \exp(xy)$. Calculate the directional derivative of F at $(1, 2)$ in the direction θ.
6. Let $F(x, y) = \exp(xy)$. Calculate the derivative at 1 of the function f defined on a neighbourhood of 1 by

   $$y = f(x) \Leftrightarrow F(x, y) = e^2$$

   where (x, y) is in a neighbourhood of $(1, 2)$.

Solutions

1.



Level curves of $\log|z| = k$

Level curves of $\operatorname{Arg} z = k$

Fig. 3

2. See page 7.

3. $\dfrac{\partial F}{\partial x}(x, y) = \cos(x^2 + y) \cdot 2x$ and $\dfrac{\partial F}{\partial y}(x, y) = \cos(x^2 + y) \cdot 1$, using the Chain Rule.

4. Let $f = U + iV$; then

$$U(x, y) = \cos x \cosh y, \quad V(x, y) = -\sin x \sinh y.$$

Now

$$\frac{\partial U}{\partial x}(x, y) = -\sin x \cosh y = \frac{\partial V}{\partial y}(x, y),$$

$$\frac{\partial V}{\partial x}(x, y) = -\cos x \sinh y = -\frac{\partial U}{\partial y}(x, y).$$

Since the partial derivatives are continuous, by the Analyticity Criterion f is analytic on **C**.

5. $\dfrac{\partial F}{\partial x}(x, y) = \exp(xy) \cdot y$, $\dfrac{\partial F}{\partial y}(x, y) = \exp(xy) \cdot x$, and so

$$\nabla_\theta F(1, 2) = e^2 \cdot 2 \cdot \cos\theta + e^2 \cdot 1 \cdot \sin\theta = e^2(2\cos\theta + \sin\theta).$$

6. We use the intermediate results of Self-Assessment Question 5. Thus

$$f'(1) = -\frac{\partial F}{\partial x}(1, 2) \Big/ \frac{\partial F}{\partial y}(1, 2) = -2/1 = -2.$$

16.2 HARMONIC FUNCTIONS

In this section we shall begin to study Laplace's equation. To do this we shall interpret much of Sections 9.1 and 9.2 of *Unit 9, Cauchy's Theorem II*, in the context of real-valued functions. To begin with, we make the crucial definition of this unit.

Definition

Let S be a region of the complex plane. A real-valued function U defined on S is called **harmonic** on S if

(a) $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0$ on S, and

(b) the second partial derivatives of U are continuous on S.

For reasons of history and convenience, we often abbreviate the expression $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}$ to $\nabla^2 U$ – the symbols ∇^2 are pronounced "del-squared"; we shall see more of ∇^2 later. The equation $\nabla^2 U = 0$ is called **Laplace's equation** for U.

In the following example we look at two functions which satisfy Laplace's equation and also introduce an abbreviated form of notation for partial derivatives.

Example 1

(a) Let $U(x, y) = x^2 - y^2$, that is $U:(x, y) \longrightarrow x^2 - y^2$. Then $\dfrac{\partial U}{\partial x}:(x, y) \longrightarrow 2x$ and $\dfrac{\partial^2 U}{\partial x^2}: (x, y) \longrightarrow 2$. We abbreviate this to $\dfrac{\partial^2 U}{\partial x^2} = \dfrac{\partial}{\partial x}(2x) = 2$. Using this notation, we have $\dfrac{\partial^2 U}{\partial y^2} = \dfrac{\partial}{\partial y}(-2y) = -2$; so that $\nabla^2 U = 0$.

Also $\dfrac{\partial^2 U}{\partial x \partial y} = \dfrac{\partial^2 U}{\partial y \partial x} = 0$, so that all the second partial derivatives of U are continuous. Hence U is harmonic on $\mathbf{C}$.

(b) Let $V(x, y) = \sin x \cosh y$. Then

$$\frac{\partial^2 V}{\partial x^2} = -\sin x \cosh y, \text{ and } \frac{\partial^2 V}{\partial y^2} = \sin x \cosh y,$$

so that $\nabla^2 V = 0$; thus V is harmonic on $\mathbf{C}$ (since again the second partial derivatives are continuous).

Basic Theorems

It may have struck you in Example 1 that $U(x, y) = \mathrm{Re}\big((x + iy)^2\big)$ and that $V(x, y) = \mathrm{Re}\big(\sin(x + iy)\big)$. In fact these examples are typical; as the next two theorems will show, the set of harmonic functions is the same as the set of real parts of analytic functions. First, we prove the easier direction of this.

Theorem 1

Let f be analytic on a region S, and $U = \mathrm{Re}\, f$. Then U is harmonic on S.

Proof

If g is an analytic function, then its real part $U_1 = \operatorname{Re} g$ and imaginary part $V_1 = \operatorname{Im} g$ satisfy the Cauchy–Riemann equations

$$\frac{\partial U_1}{\partial x} = \frac{\partial V_1}{\partial y} \quad \text{and} \quad \frac{\partial U_1}{\partial y} = -\frac{\partial V_1}{\partial x}. \tag{1}$$

We want to end up with second derivatives, so a good move might be to take g to be f'; note that by analyticity of derivatives, g is analytic on S.

Now, since $f = U + iV$,

$$g = f' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

(by the Cauchy–Riemann equations for f),

and so, by the definition of U_1 and V_1,

$$U_1 = \frac{\partial U}{\partial x}, \quad V_1 = -\frac{\partial U}{\partial y}. \tag{2}$$

Thus,

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2} &= \frac{\partial U_1}{\partial x}, \quad \text{by (2),}\\ &= \frac{\partial V_1}{\partial y}, \quad \text{by (1),}\\ &= -\frac{\partial^2 U}{\partial y^2}, \quad \text{by (2),}\end{aligned}$$

and so $\nabla^2 U = 0$ on S. Also, since g is analytic, its partial derivatives are continuous and so the second partial derivatives of f are continuous on S. Hence, U is harmonic on S. ■

It follows by a similar argument that $V = \operatorname{Im} f$ is harmonic on S; alternatively, we can deduce this from Theorem 1 and the observation that $V = \operatorname{Re}(-if)$, since, of course, the function $-if$ is also analytic on S.

Using this theorem gives us quicker proofs of the results in Example 1, and many similar results can be derived. For example, if S is the cut plane $\mathbf{C} - \{t \in \mathbf{R} : t \leqslant 0\}$ then Log is analytic on S, and so $U = \operatorname{Re} \operatorname{Log}$ is harmonic on S; it is easy to calculate that

$$U(x, y) = \log \sqrt{x^2 + y^2},$$

that is, $U(z) = \log |z|$.

In fact, U is harmonic not just on the cut plane S, but actually on the punctured plane $\mathbf{C} - \{0\}$, as can be verified by direct calculation—see Problem 1 of Section 16.3. But on $\mathbf{C} - \{0\}$, U does not seem to be of the form $\operatorname{Re} f$ for any analytic f—again, see Problem 1 of Section 16.3. Thus it seems that the obvious converse of Theorem 1 (that is, a harmonic function is the real part of some analytic function, on any given region) is false, and so we must be a little more careful. Not surprisingly, perhaps, simple-connectedness comes to our rescue again, and we can prove the following theorem.

Theorem 2

Let U be harmonic on a simply-connected region S. Then there exists a function f analytic on S such that $U = \operatorname{Re} f$ on S.

Proof

We establish this by reversing the steps of Theorem 1, using appropriate results to push this through.

Let $g = \dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$. Let $U_1 = \operatorname{Re} g$ and $V_1 = \operatorname{Im} g$. We first show that U_1 and V_1 satisfy the Cauchy–Riemann equations.

$$\begin{aligned}\frac{\partial U_1}{\partial x} &= \frac{\partial^2 U}{\partial x^2}, \quad \text{by the definition of } U_1,\\ &= -\frac{\partial^2 U}{\partial y^2}, \text{ since } U \text{ is harmonic,}\\ &= \frac{\partial}{\partial y}\left(-\frac{\partial U}{\partial y}\right)\\ &= \frac{\partial V_1}{\partial y}, \quad \text{by the definition of } V_1.\end{aligned}$$

Also

$$\begin{aligned}\frac{\partial U_1}{\partial y} &= \frac{\partial^2 U}{\partial y \partial x},\\ &= \frac{\partial^2 U}{\partial x \partial y}, \quad \text{by the Commuting Theorem in Section 16.1,}\\ &= -\frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial y}\right)\\ &= -\frac{\partial V_1}{\partial x}.\end{aligned}$$

Since the partial derivatives of U_1 and V_1 are continuous, because they are essentially the second partial derivatives of U, the function $g = U_1 + iV_1$ is analytic on S (by the Analyticity Criterion of Section 16.1). But since S is simply-connected, the Antiderivative Theorem of *Unit 9* tells us that there is a function f_2 analytic on S such that $f_2' = g$.

Let $U_2 = \operatorname{Re} f_2$, $V_2 = \operatorname{Im} f_2$. Then

$$\frac{\partial U_2}{\partial x} - i\frac{\partial U_2}{\partial y} = f_2' = g = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y},$$

and so $\dfrac{\partial}{\partial x}(U_2 - U) = \dfrac{\partial}{\partial y}(U_2 - U) = 0$. Hence, by the Constancy Condition of Section 16.1, $U_2 - U = c$ for some constant c. Let $f = f_2 - c$: then $\operatorname{Re} f = U_2 - c = U$. ■

Even if the region S is not simply-connected, we can still obtain some useful information from Theorem 2, which is best stated as a corollary.

Corollary

Let U be harmonic on a region R. Then for each point $z_0 \in R$ there is a function f analytic on a neighbourhood D of z_0 such that $U = \operatorname{Re} f$ on D.

Proof

Let D be an open disc centre z_0 contained in R (note that R is open). Since D is simply-connected, we can apply Theorem 2 to U and $S = D$. ■

Thus the function U defined by $U(z) = \log|z|$ is of the form $\operatorname{Re} f$ on a neighbourhood of any point $z_0 \neq 0$ (take f to be a branch of the logarithm whose domain contains z_0; note that if z_0 is real and negative, then f cannot be chosen to be Log).

Arising out of this last theorem is our next definition.

Definition

> Let U and V be functions harmonic on a region S. We call V a **harmonic conjugate of U on S** if $U + iV$ is analytic on S.

Equivalently, V is a harmonic conjugate of U if the Cauchy–Riemann equations hold for U and V, that is

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \text{ on } S.$$

Note that if V is a harmonic conjugate of U, so is $V + c$ for any real constant function c. There is also a kind of symmetry between harmonic conjugates—if V is a harmonic conjugate of U, then $-U$ is a harmonic conjugate of V—see Self-Assessment Question 5.

Basically, there are two ways of finding harmonic conjugates. The first uses the method of Theorem 2. The next example illustrates both methods.

Example 2

Let $U(x, y) = x^3 - 3xy^2 + y$. Show that U is harmonic on **C** and find a harmonic conjugate for U.

First Solution

We have

$$\frac{\partial U}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 U}{\partial x^2} = 6x, \quad \frac{\partial^2 U}{\partial y \partial x} = -6y,$$

$$\frac{\partial U}{\partial y} = -6xy + 1, \quad \frac{\partial^2 U}{\partial y^2} = -6x, \quad \frac{\partial^2 U}{\partial x \partial y} = -6y.$$

Thus $\nabla^2 U = 0$, and the second partial derivatives of U are continuous, so that U is harmonic on **C**.

Let $g = \dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$. We know that g is entire (by Theorem 2).

Now

$$\begin{aligned} g(x + iy) &= 3x^2 - 3y^2 - i(-6xy + 1) \\ &= 3x^2 + 6ixy - 3y^2 - i \\ &= 3(x + iy)^2 - i, \end{aligned}$$

that is, $g(z) = 3z^2 - i$. Now by Theorem 2, there is an entire function f such that $f'(z) = g(z) = 3z^2 - i$. Integrating, we obtain $f(z) = z^3 - iz + \alpha + i\beta$, for some real constants α and β. Now $(\text{Re} f)(x, y) = x^3 - 3xy^2 + y + \alpha$. Since we want $U = \text{Re} f$, we choose $\alpha = 0$. The required harmonic conjugate of U is $V = \text{Im} f$. Now $V(x, y) = 3x^2y - y^3 - x + \beta$.

To summarize: we express $\dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$ as a function g of a complex variable, integrate g to get f, adjust the constant to match U with $\text{Re} f$, and then read off $\text{Im} f$, which is the required harmonic conjugate.

The second method does not explicitly mention complex numbers at all; it solves for V using the Cauchy–Riemann equations.

Second Solution

Suppose that V is a harmonic conjugate of U. Then, since $\dfrac{\partial V}{\partial y} = \dfrac{\partial U}{\partial x}$,

$$\frac{\partial V}{\partial y}(x, y) = 3x^2 - 3y^2.$$

Now we "integrate both sides with respect to y", as follows.

Choose any fixed x in $\mathbf{R}$. In terms of the one variable, y, we have

$$(y \longrightarrow V(x, y))' = (y \longrightarrow 3x^2 - 3y^2) = (y \longrightarrow 3x^2y - y^3)'.$$

So the functions $y \longrightarrow V(x, y)$ and $y \longrightarrow 3x^2y - y^3$ differ by a constant, which may depend on x and so we denote it by $c(x)$. That is $V(x, y) = 3x^2y - y^3 + c(x)$, for all $y \in \mathbf{R}$. This holds for each $x \in \mathbf{R}$. But also $-\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}$, so that, for each fixed $y \in \mathbf{R}$, $6xy - 1 = 6xy + c'(x)$. (Note that $x \longrightarrow c(x)$ is differentiable, since it is the difference of two differentiable functions, $x \longrightarrow V(x, y)$ and $x \longrightarrow 3x^2y - y^3$.) Thus, $c'(x) = -1$, and so $c(x) = -x + \beta$ for some real constant β. Then

$$V(x, y) = 3x^2y - y^3 - x + \beta.$$

Now, by reversing these steps, it is clear that $\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$, so that this V is indeed a harmonic conjugate of U.

There is one other direct consequence of Theorems 1 and 2 that should be mentioned.

A harmonic function has all partial derivatives of all orders.

The proof of this result proceeds by induction on the order of the partial derivatives and use of the Cauchy–Riemann Equations, but we shall not give the proof.

Properties of Harmonic Functions

Because of the close correspondence between analytic and harmonic functions, we can translate several of the results of *Unit 5, Cauchy's Theorem I* and *Unit 9, Cauchy's Theorem II*, into results about harmonic functions. The two most important results are the Boundary Uniqueness Theorem, which tells us that a harmonic function is determined by its boundary values, and the analogue of Cauchy's Formula, which tells us, in some cases, how to find it. We begin by working towards the Boundary Uniqueness Theorem.

Theorem 3 (The Local Maximum Principle for Harmonic Functions)

Let R be a region, U a function harmonic on R which has a local maximum at a point α in R. Then U is constant on a neighbourhood of α.

Proof

This is very close to the solution of Problem 3 in Section 9.2 of *Unit 9*. By Theorem 2, there is an open disc D with centre α in R and a function f analytic on D such that $U = \operatorname{Re} f$. Now e^f is analytic on D and $|e^f|$, which equals e^U, has a maximum on D at α. Hence e^f is constant on D (by the Local Maximum Principle, Theorem 1 of *Unit 9*). Thus e^U, which is $|e^f|$, is constant on D, and so U is constant on D. ■

Theorem 4 (The Maximum Principle for Harmonic Functions)

Let R be a bounded region, U a function harmonic on R and continuous on $\bar{R}$, the closure of R. Let M be the maximum value of U on $\bar{R}$. Then $U(z) < M$ for all $z \in R$, unless U is constant on R. Thus the maximum value of U is attained on the boundary of R.

Proof

Certainly $U(z) \leqslant M$ for all $z \in R$. Now suppose that $U(\alpha) = M$ for some $\alpha \in R$. Let $G_1 = \{z \in R : U(z) \neq M\}$, $G_2 = \{z \in R : U(z) = M\}$. Then G_1 is open, $G_1 \cup G_2 = R$ and $G_1 \cap G_2 = \varnothing$. Also G_2 is open; for if $U(z) = M$, then U has a local maximum at z, and so by Theorem 3, U is constant, with value M, on a neighbourhood of z. Since R is connected and $G_2 \neq \varnothing$ (because $\alpha \in G_2$), $G_2 = R$: that is $U(z) = M$ for all $z \in R$. ■

From Theorem 4 we can now deduce a uniqueness theorem analogous to the Boundary Uniqueness Theorem of *Unit 9*, in the manner of the solution to Problem 5 of Section 9.2 there.

Theorem 5 (The Boundary Uniqueness Theorem for Harmonic Functions)

Let U_1 and U_2 be harmonic on a bounded region R, and continuous on $\bar{R}$. If $U_1 = U_2$ on the boundary of R, then $U_1 = U_2$ on R.

Proof

Let $U = U_1 - U_2$. It is easy to check that U is harmonic on R (see Problem 2 of the next section). Also U is continuous on $\bar{R}$, and $U = 0$ on the boundary of R. By Theorem 4, the maximum value of U is attained on the boundary, so that $U(z) \leqslant 0$ for all $z \in R$. Similarly, the minimum value of U (that is, the maximum value of $-U$) is attained on the boundary, and so $U(z) \geqslant 0$ for all $z \in R$. Thus U which is harmonic on R and satisfies certain conditions on the boundary of R?

This last theorem serves to introduce the major concern of this unit, solving Laplace's equation. We can state this in the form of a problem.

The General Boundary Value Problem Given a region R, can we find a function U which is harmonic on R and satisfies certain conditions on the boundary of R?

This problem is too vaguely formulated for us to solve, so we have to be more restrictive. In particular, if these conditions say that U has to be equal to some specified function on the boundary and continuous at "most" points on the boundary, then we call the problem of finding such a U a **Dirichlet problem**, but even this is not precise enough. (Unlike most of the pure mathematics you have done, in mathematics with applications it is often difficult to give precise but all-embracing definitions.) So here is a special kind of Dirichlet problem, which we *can* state precisely.

The Dirichlet Problem—a special case Let R be the inside of a simple-closed contour Γ, and let $U_0 : \Gamma \longrightarrow \mathbf{R}$ be a real-valued function continuous on Γ. Find a function U harmonic on R and continuous on $\bar{R} = R \cup \Gamma$ such that $U = U_0$ on Γ.

There are theoretical results which show that such a function U must *exist*; however, we may have great difficulty in *constructing* one. This problem is a very important one in applied mathematics; as we shall see in Section 16.7, the harmonic function U usually provides an answer to some physical problem. Because of this, it is important to prove that U is unique. The *physical* importance of Theorem 5 is that it guarantees the uniqueness of the solution U to the special case of Dirichlet's problem formulated above. The *mathematical* importance is that if we can guess a solution which satisfies the boundary conditions then it is the only solution.

Unfortunately, Theorem 5 does not help us to *find* a solution, and this is the problem we want to discuss now. By and large, we would expect that the "nicer" a region R is, the easier it will be to find U. So let us concentrate on the nicest sort of region, an open disc.

Theorem 6 (Poisson's Formula for a Disc)

Let C be the circle $|z| = r$ and D the open disc $|z| < r$. Suppose that U_0 is a real-valued function continuous on C. Then there is a unique function U harmonic on D and continuous on $\bar{D}$, with $U = U_0$ on C, and it is defined in D by

$$U(\rho e^{i\phi}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{U_0(re^{i\theta})}{r^2 - 2r\rho\cos(\theta - \phi) + \rho^2}\, d\theta, \quad \rho < r. \tag{3}$$

(You may have seen this formula before, if you have studied *Unit M321 10, Green's Functions II: Partial Differential Equations.*)

We shall not be able to give the full proof of this result, as it would take too long, but we *shall* prove some bits. Note first that uniqueness follows by Theorem 5. Next let us establish a lemma.

Lemma

Let C be the circle $|z| = r$ and D the open disc $|z| < r + \delta$, where $\delta > 0$. Suppose also that U is harmonic on D. Then

$$U(\rho e^{i\phi}) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{U(re^{i\theta})}{r^2 - 2r\rho\cos(\theta - \phi) + \rho^2}\, d\theta, \quad \rho < r.$$

Proof

Let f be analytic on D with $U = \operatorname{Re} f$ (by Theorem 2). Let $\rho < r$, $\phi \in [0, 2\pi]$, and $z = \rho e^{i\phi}$.

By Cauchy's Formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We now want to take real parts, as the real part of the left-hand side is just $U(\rho e^{i\phi})$. However, the real part of the right-hand side is messy at the moment. We simplify it by the amusing trick of subtracting an integral with value 0 from it.

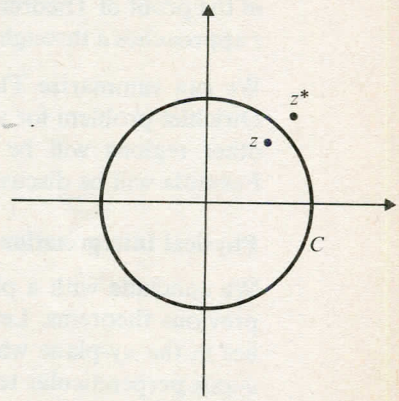


Fig. 4

Specifically, let $z^* = r^2/\bar{z}$. We call z^* the *reflection of z in the circle C*. (See *Unit M100 29, Complex Numbers II*, Section 29.3.1.) Now $|z^*| = r^2/|z| > r$, and so z^* lies outside C (Fig. 4), so that by Cauchy's Theorem,

$$0 = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z^*}\, d\zeta.$$

Hence,

$$f(z) = f(z) - 0 = \frac{1}{2\pi i} \int_C f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*}\right) d\zeta.$$

Now for $\zeta \in C$ (that is, $|\zeta| = r$),

$$\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} = \frac{1}{\zeta - z} - \frac{\bar{z}}{\zeta(\bar{z} - \bar{\zeta})}, \quad \text{since } \zeta\bar{\zeta} = r^2 \text{ and } z^* = r^2/\bar{z},$$

$$= \frac{-\zeta\bar{z} + \zeta\bar{\zeta} + \zeta\bar{z} - z\bar{z}}{\zeta|\zeta - z|^2} = \frac{r^2 - |z|^2}{\zeta|\zeta - z|^2}.$$

Hence,

$$f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \frac{r^2 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{\zeta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})(r^2 - \rho^2)}{|re^{i\theta} - \rho e^{i\phi}|^2}\, d\theta, \tag{4}$$

using the substitution $\zeta = re^{i\theta}$, $d\zeta = ire^{i\theta}\, d\theta$.

Now

$$\begin{aligned}|re^{i\theta} - \rho e^{i\phi}|^2 &= (re^{i\theta} - \rho e^{i\phi})(re^{-i\theta} - \rho e^{-i\phi})\\ &= r^2 + \rho^2 - r\rho(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)})\\ &= r^2 + \rho^2 - 2r\rho\cos(\theta - \phi).\end{aligned}$$

Thus, by taking real parts of (4),

$$U(\rho e^{i\theta}) = \frac{r^2 - \rho^2}{2\pi}\int_0^{2\pi} \frac{U(re^{i\theta})}{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}\, d\theta,$$

which is the required formula. ■

By using a uniform continuity argument rather like that for Cauchy's Formula in *Unit 5* (see Problem 6 of Section 5.8), we can weaken the hypotheses of the lemma, by taking $\delta = 0$, provided we assume that U is continuous on $D \cup C$. Thus if U *exists*, and is harmonic on D and continuous on $\bar{D}$, then U is given by Poisson's Formula (3) in the statement of Theorem 6 (with $U_0(z) = U(z)$, $z \in C$).

Now suppose that U_0 is a real-valued continuous function defined only on the boundary C of D. It is not very hard to prove that the function U defined on D by Poisson's Formula (3) is harmonic on D. The remaining, and difficult, part of the proof of Theorem 6 is to show that when a lies on C, the limit of $U(z)$ as z approaches a through D is equal to $U_0(a)$, and we shall not discuss this further.

We can summarize Theorem 6 by saying that Poisson's Formula solves the Dirichlet problem for an open disc. Methods of solving Dirichlet problems for other regions will be discussed in Section 16.7. Examples using Poisson's Formula will be discussed also in Section 16.7.

Physical Interpretation

We conclude with a physical model which may help to illustrate some of the previous theorems. Let R be the inside of a simple-closed contour Γ. Then R lies in the xy-plane which we shall imagine as being horizontal. Now add the u-axis perpendicular to the xy-plane.

Let $U_0:\Gamma \longrightarrow \mathbf{R}$ be a real-valued function. Draw its graph, that is, the set $S = \{(x, y, u):(x, y) \text{ on } \Gamma \text{ and } u = U_0(x, y)\}$. Now S lies directly above Γ and will look something like that in Fig. 5. (We assume for simplicity that $U_0 > 0$.)

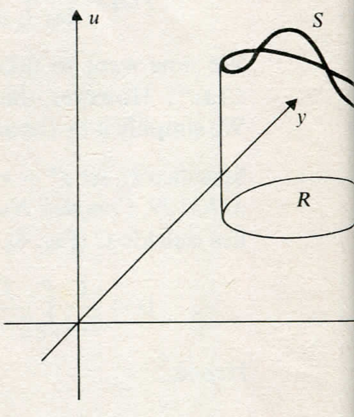


Fig. 5

Then we can find the function U as follows: make a piece of bent wire into the exact shape of S and support it rigidly above the contour Γ in the xy-plane at the same height as S is. In other words, "solidify" the abstract line S into a line of metal. Dip S in a bowl of soapy water, and remove it again. With a little bit of skill, you will be able to form a soap film stretched across the twisted wire S, and you will always get the same one. We naturally define U by $U(x, y) =$ height of the point on the soap film above (x, y). (Can U have a local maximum or minimum?)

It can be shown that U is a good approximation to a harmonic function, and it clearly satisfies the boundary conditions. How about that as a way of proving that U always exists, and is unique?

Summary

This section split into two main parts. In the first, we defined the notion of harmonic function, and showed that, locally at least, harmonic functions are precisely the real parts of analytic functions. A particular consequence of this was the existence of harmonic conjugates.

In the second part, we discussed the properties of harmonic functions, the fundamental result being the Maximum Principle. This led on to the Boundary Uniqueness Theorem, that a function harmonic on a region is determined by its values on the boundary, and then to a discussion of the Dirichlet problem and Poisson's Formula.

We concluded with a model of harmonic functions given by soap films.

Self-Assessment Questions

1. What is Laplace's equation, and what has it to do with harmonic functions?
2. Why is the function $(x, y) \longrightarrow \mathrm{Re}(\cosh(x + iy))$ harmonic on **C**?
3. Suppose that U is a real-valued function harmonic on the open disc $D = \{z : |z| < 1\}$. Are there any functions f analytic on D of special relevance to U?
4. Define "harmonic conjugate".
5. Let V be a harmonic conjugate of U. Show that $-U$ is a harmonic conjugate of V.
6. State the Maximum Principle for Harmonic Functions.
7. What is the Dirichlet problem? What theorems in this section are relevant to it?

Solutions

1. Laplace's equation for a function U on a region S states that $\nabla^2 U = 0$ on S. A function U is harmonic on S if it satisfies Laplace's equation *and* has continuous second partial derivatives on S.
2. By Theorem 1, since the function is the real part of cosh, which is analytic on **C**.
3. Yes: by Theorem 2 there is a function f analytic on D such that $U = \mathrm{Re}\, f$.
4. See page 16.
5. If $U + iV$ is analytic, so is $-i(U + iV) = V + i(-U)$.
6. See page 17.
7. The Dirichlet Problem is discussed on page 18. Theorem 5 (the Boundary Uniqueness Theorem) and Theorem 6 (Poisson's Formula) are especially relevant to it; the first establishes uniqueness of a solution (under certain circumstances), and the second exhibits a formula giving a solution for a disc.

16.3 PROBLEMS

1. Let $U(z) = \log|z|$ and $R = \{z : z \in \mathbf{C}, z \neq 0\}$.

   (a) Show that U is harmonic on the punctured plane R.

   (b) Show that there exists no function f analytic on R such that $U = \operatorname{Re} f$.

2. Let U and V be harmonic on a region S, and α, β be real numbers. Show that $\alpha U + \beta V$ is harmonic on S. (This simple result is very useful, as you will see in Section 16.7.)

3. A result which will be very useful in Section 16.7 concerns the behaviour of harmonic functions under composition with analytic functions.

   Let $U : S \longrightarrow \mathbf{R}$ be harmonic on the region S, and $g : R \longrightarrow S$ be analytic on the region R. Show that $U \circ g : R \longrightarrow \mathbf{R}$ is harmonic on R. (Hint: Consider composition of analytic functions.)

4. Find harmonic conjugates of the following harmonic functions on the given regions.

   (i) $U_1(x, y) = \sin x \sinh y, \quad R_1 = \mathbf{C}.$

   (ii) $U_2(x, y) = \log|x + iy|, \quad R_2 = \mathbf{C} - \{t : t \in \mathbf{R}, t \leqslant 0\}.$

   (iii) $U_3(x, y) = xe^x \cos y - ye^x \sin y, \quad R_3 = \mathbf{C}.$ (Use the method of the second solution to Example 2 on page 16, unless you can do it quickly.)

5. Let R be a bounded region, and U, V be functions harmonic on R and continuous on $\bar{R}$ such that $U \leqslant V$ on the boundary B of R. Prove that $U \leqslant V$ on R. (What does this mean in terms of the soap film interpretation on page 20?)

Solutions

1. (a) We have $U(x, y) = \log\sqrt{x^2 + y^2} = \frac{1}{2}\log(x^2 + y^2)$. Thus

$$\frac{\partial U}{\partial x} = \frac{1}{2}\cdot\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2},$$

and

$$\frac{\partial U}{\partial y} = \frac{y}{x^2 + y^2}, \quad \text{by symmetry.}$$

Hence,

$$\frac{\partial^2 U}{\partial x^2} = \frac{(x^2 + y^2)\cdot 1 - x\cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad \text{and} \quad \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} = -\frac{2xy}{(x^2 + y^2)^2}.$$

Thus $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0$. Also the second partial derivatives are clearly continuous, and so U is harmonic on R.

Alternatively, this follows from the remark after the proof on page 14.

(b) Assume that there exists a function f analytic on $R = \{z : z \in \mathbf{C}, z \neq 0\}$ such that $U = \operatorname{Re} f$, where $U(z) = \log|z|$. Then $f = \operatorname{Log}$ on $\{z : z \text{ is real and positive}\}$, and so by the Uniqueness Theorem of *Unit 6*, $f = \operatorname{Log}$ on $R_1 = \{z : z \in \mathbf{C}, z \text{ is not zero or real and negative}\}$. Thus the function element (f, R) is a direct analytic continuation of $(\operatorname{Log}, R_1)$. But from *Unit 11*, Section 11.5, we know that $(\operatorname{Log}, R_1)$ cannot be continued analytically onto R, and so we have a contradiction. Thus no such analytic function f exists.

(An alternative to using the Uniqueness Theorem, is to use Theorem 17 of *Unit 3*.)

2. The operators of partial differentiation are *linear*, just like the one-variable differentiation operators. Thus

$$\frac{\partial}{\partial x}(\alpha U + \beta V) = \alpha\frac{\partial U}{\partial x} + \beta\frac{\partial V}{\partial x}, \text{ and so on.}$$

Hence, if $\nabla^2 U = \nabla^2 V = 0$,

$$\nabla^2(\alpha U + \beta V) = \alpha\nabla^2 U + \beta\nabla^2 V = 0 + 0 = 0.$$

3. Let $\alpha \in R$. Then $(U \circ g)(\alpha) = U\big(g(\alpha)\big)$, and so we should look at the behaviour of U at $g(\alpha)$.

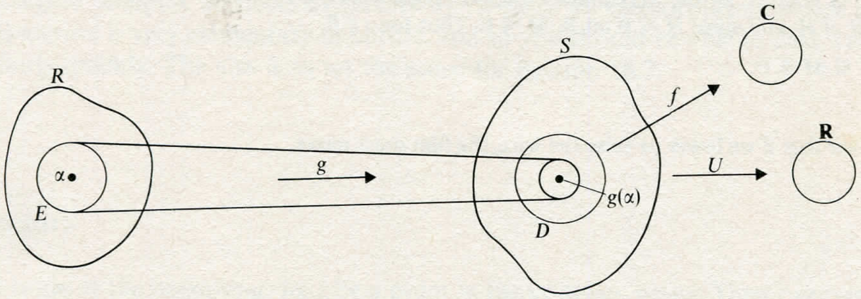


Fig. 6

Now S may not be simply-connected, but at least (by the corollary to Theorem 2) there is an open disc D centre $g(\alpha)$ contained in S and a function f analytic on D such that $U = \operatorname{Re} f$. Thus $U \circ g = \operatorname{Re}(f \circ g)$; but we must find a suitable region on which $f \circ g$ is defined. Since g is continuous at α, there is an open disc E centre α such that $g(E) \subseteq D$ (Fig. 6). Then $f \circ g$ is defined on E; furthermore, $f \circ g$ is analytic on E. Hence by Theorem 1, $U \circ g = \operatorname{Re}(f \circ g)$ is harmonic on E, that is, on a neighbourhood of α. Since this holds for each $\alpha \in R$, $U \circ g$ is harmonic on R.

4. (i) The best way of doing this is to dredge up vague memories of trigonometric addition formulas.

Now

$$\begin{aligned}\cos(x + iy) &= \cos x \cos iy - \sin x \sin iy\\ &= \cos x \cosh y - i \sin x \sinh y.\end{aligned}$$

Thus $U_1(x, y) = \operatorname{Re}\big(i\cos(x + iy)\big)$, and so U_1 is harmonic on $\mathbf{C}$. Thus a harmonic conjugate of U_1 is $V_1(x, y) = \operatorname{Im}\big(i\cos(x + iy)\big) = \cos x \cosh y$. For an alternative method, see the style of solution to part (iii).

(ii) Since $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$, $z \in R_2$, a harmonic conjugate of $U_2(z) = \log|z|$ is Arg. (Thus, in particular, Arg is harmonic on R_2.)

(iii) Let V be a harmonic conjugate of U_3. Thus

$$\begin{aligned}\frac{\partial V}{\partial y}(x, y) = \frac{\partial U_3}{\partial x}(x, y) &= e^x \cos y + xe^x \cos y - ye^x \sin y\\ &= \cos y(e^x + xe^x) - e^x y \sin y.\end{aligned}$$

Integrating with respect to y, as in the second solution to Example 2 of Section 16.2, we get

$$V(x, y) = \sin y(e^x + xe^x) - e^x \int y \sin y \, dy + c(x).$$

Now $\int y \sin y\, dy = -y\cos y + \sin y$, so that

$$\begin{aligned}V(x, y) &= \sin y(e^x + xe^x) - e^x(\sin y - y\cos y) + c(x)\\ &= xe^x \sin y + ye^x \cos y + c(x).\end{aligned}$$

We must now determine the function $x \longrightarrow c(x)$. We have

$$-\frac{\partial U_3}{\partial y} = \frac{\partial V}{\partial x},$$

so that

$$\begin{aligned}&-\big(xe^x(-\sin y) - e^x \sin y - ye^x \cos y\big)\\ &\quad = e^x \sin y + xe^x \sin y + ye^x \cos y + c'(x);\end{aligned}$$

and so, simplifying, $0 = c'(x)$, from which $c(x) = k$ for some constant k. We have only to find one harmonic conjugate so we may as well take $k = 0$, so that

$$V(x, y) = xe^x \sin y + ye^x \cos y$$

is a harmonic conjugate of U_3.

Of course, you may have bypassed this process by "observing" (after a little discrete manipulation) that $U_3(x, y) = \text{Re}(ze^z)$ where $z = x + iy$, and hence finding V.

5. Consider $H = U - V$: then H is harmonic on R (by Problem 2) and continuous on $\bar{R}$. Let M be the maximum value of H on $\bar{R}$. By the Maximum Principle (Theorem 4), M is attained on the boundary B of R, and since $H \leqslant 0$ on B, $M \leqslant 0$. Thus for $z \in R$,

$$U(z) - V(z) = H(z) \leqslant M \leqslant 0;$$

that is, $U \leqslant V$ on R.

(In terms of the soap film, we see that if we lower or raise the wire, the film must move with it.)

16.4 VECTOR FIELDS

This section provides an introduction to the topic of *vector analysis*. Facility with vectors is crucial if you are to become a good applied mathematician. You may well have studied part of the subsequent material in courses such as MST282, Mechanics and Applied Calculus. However, if you have not, do not worry: we are going to deal exclusively with vectors in the plane, and in this case, we can use complex analysis to give a self-contained treatment. (In fact, we regard complex numbers as vectors by virtue of an isomorphism.) Much of this section is very elementary in nature, and so you should be able to deal with it fairly quickly. The aim is to set the scene for Section 16.7.

Vectors

To begin at the beginning: let z be a point in the complex plane. Then z corresponds to a vector, namely the geometric vector $\underset{\rightarrow}{Oz}$. For reasons of convenience and typography, we shall write $\mathbf{z}$ for $\underset{\rightarrow}{Oz}$.

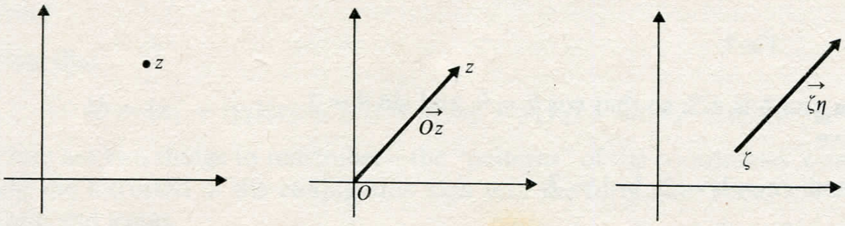


Fig. 7

In this case $\mathbf{z}$ is represented by an arrow $\overrightarrow{Oz}$ from the origin O, but as you may recall from *Unit M100 22, Linear Algebra I*, we can represent $\mathbf{z}$ by an arrow of the same length and direction from any point, such as $\overrightarrow{\zeta\eta}$ in Fig. 7. As you also may recall from *Unit M100 27, Complex Numbers I*, addition of complex numbers corresponds to addition of the corresponding geometric vectors. In fact, the function $J:\mathbf{z} \longrightarrow z$ is an *isomorphism* from the vector space G^2 of geometric vectors in the plane onto the set $\mathbf{C}$ of complex numbers regarded as a vector space. Under this isomorphism there is a correspondence between many vector and complex number concepts: for example, the length $|\mathbf{z}|$ of the vector $\mathbf{z}$ is equal to the modulus $|z|$ of the complex number z. More importantly, concepts initially defined for vectors can be "transferred" to $\mathbf{C}$ by means of the isomorphism J and so fairly soon we shall abandon the *explicit* use of vectors, even though we are using concepts of vectorial origin.

As well as addition, there are two other binary operations on plane vectors that are important in applied mathematics—but they are not *closed* since each maps pairs of vectors to real numbers.

Definition

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in G^2.

(1) The **inner product** of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \bullet \mathbf{b}$ (where $\bullet$ is read as "dot"), is defined to be $|\mathbf{a}|\,|\mathbf{b}| \cos \theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, measured anticlockwise from $\mathbf{a}$ to $\mathbf{b}$ as shown in Fig. 8.

(2) The **outer product** of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \wedge \mathbf{b}$ (where $\wedge$ is read "wedge"), is defined to be $|\mathbf{a}|\,|\mathbf{b}| \sin \theta$.

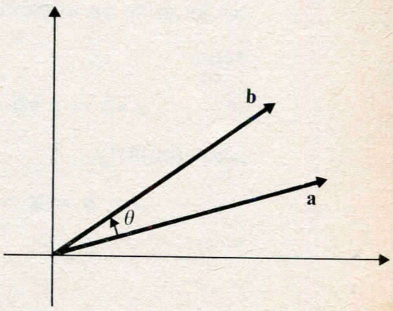


Fig. 8

Note that the angle θ can be taken to be $\psi - \phi$, where ψ is an argument of $\mathbf{b}$ and ϕ an argument of $\mathbf{a}$.

(You may have come across the *vector product* $\mathbf{a} \times \mathbf{b}$ in other courses; for example MST 282, Mechanics and Applied Calculus. Our operation $\wedge$ is closely related to $\times$. In fact, if the three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ happen to lie in the xy-plane, then $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the xy-plane whose length is $|\mathbf{a} \wedge \mathbf{b}|$.)

Example 1

Calculate $\mathbf{a} \bullet \mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{b} \bullet \mathbf{a}$ and $\mathbf{b} \wedge \mathbf{a}$ when $\mathbf{a} = (2, 0)$ and $\mathbf{b} = (0, 1)$.

Solution

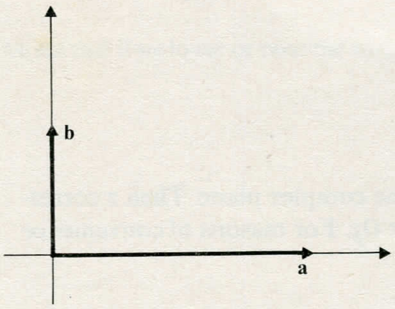


Fig. 9

Note that the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/2$, so that $\cos\theta = 0$, and $\sin\theta = 1$. Since $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 1$, we have

$$\mathbf{a} \bullet \mathbf{b} = 2 \cdot 1 \cdot 0 = 0 \quad \text{and} \quad \mathbf{a} \wedge \mathbf{b} = 2 \cdot 1 \cdot 1 = 2.$$

The angle between $\mathbf{b}$ and $\mathbf{a}$ is $2\pi - \theta$, so that

$$\mathbf{b} \bullet \mathbf{a} = 2\cos(2\pi - \theta) = 2\cos(-\theta) = 2\cos\theta = 0$$

and

$$\mathbf{b} \wedge \mathbf{a} = 2\sin(-\theta) = -2\sin\theta = -2.$$

In general, for nonzero $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \bullet \mathbf{b} = 0$ precisely if $\mathbf{a}$ is perpendicular to $\mathbf{b}$—see Self-Assessment Question 6 on page 35.

Under the one–one function $J: G^2 \longrightarrow \mathbf{C}$, these two operations on vectors ($\bullet$ and $\wedge$) induce two operations on complex numbers. We denote the two induced operations by the same symbols, and so $a \bullet b = \mathbf{a} \bullet \mathbf{b}$, and $a \wedge b = \mathbf{a} \wedge \mathbf{b}$. Perhaps surprisingly, there is a relation between these operations and the usual operation of multiplication on complex numbers.

Theorem 7

Let a and b be complex numbers. Then $\bar{a}b = a \bullet b + i(a \wedge b)$.

Proof

We can assume that a and b are both nonzero, as otherwise the result is obvious. So let ϕ be an argument of a and ψ an argument of b.

Now

$$a \bullet b = \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\psi - \phi),$$

and similarly,

$$a \wedge b = \mathbf{a} \wedge \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin(\psi - \phi).$$

Thus, since $|\mathbf{z}| = |z|$, we have

$$\begin{aligned} a \bullet b + i(a \wedge b) &= |a|\,|b| \cos(\psi - \phi) + i|a|\,|b| \sin(\psi - \phi) \\ &= |a|\,|b| \left(\cos(\psi - \phi) + i \sin(\psi - \phi)\right) \\ &= |a|\,|b|\, e^{i(\psi - \phi)} \\ &= |a|\, e^{-i\phi} \cdot |b|\, e^{i\psi} \\ &= \bar{a}b. \quad \blacksquare \end{aligned}$$

From this theorem we can extract a more useful formulation of $a \bullet b$ and $a \wedge b$ for complex numbers a and b, which does not mention angles or vectors, by taking real and imaginary parts. In future we shall normally use this formulation, which is given in the following corollary.

Corollary

Let $a = x_1 + iy_1$, $b = x_2 + iy_2$ (where $x_1, x_2, y_1, y_2 \in \mathbf{R}$). Then

$$a \bullet b = \operatorname{Re}(\bar{a}b) = x_1x_2 + y_1y_2, \tag{1}$$

$$a \wedge b = \operatorname{Im}(\bar{a}b) = x_1y_2 - x_2y_1. \tag{2}$$

Proof

Note that

$$\bar{a}b = (x_1 - iy_1)(x_2 + iy_2) = (x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1). \quad \blacksquare$$

There are two things to remember—the "patterns" of the coordinates x and y, and the intrusion of the conjugation sign into the formulas. These will recur again and again.

Example 2

Show that $b \bullet a = a \bullet b$ and $b \wedge a = -(a \wedge b)$.

Solution

Either observe the (a)symmetry of expressions (1) and (2) or calculate with conjugates:

$$b \bullet a = \operatorname{Re}(\bar{b}a) = \operatorname{Re}(\overline{\bar{b}a}) = \operatorname{Re}(\bar{a}b) = a \bullet b,$$

and

$$b \wedge a = \operatorname{Im}(\bar{b}a) = -\operatorname{Im}(\overline{\bar{b}a}) = -\operatorname{Im}(\bar{a}b) = -(a \wedge b).$$

Self-Assessment Question 1

Let $a = 1$, $b = -1 + i$. Determine $a \bullet b$ and $a \wedge b$ by the reformulation in the corollary. What is the angle between the vectors $\mathbf{b}$ and $\mathbf{a}$?

Solution

$$a \bullet b = 1 \cdot (-1) + 0 \cdot 1 = -1;$$

$$a \wedge b = 1 \cdot 1 - (-1) \cdot 0 = 1.$$

The angle between **b** and **a**, that is the angle from **b** to **a** (anticlockwise) is $5\pi/4$.

Vector Fields

Having seen that complex numbers correspond to vectors, we now want to move on to a similar correspondence for complex functions. Suppose that $F: S \longrightarrow \mathbf{C}$ is a complex-valued function whose domain is a region S in $\mathbf{C}$. We can interpret F as a **vector-valued function** (that is one whose range is a set of vectors), or **vector field**, as it is often called, by defining $\mathbf{F}(z)$ to be the vector corresponding to $F(z)$, for each $z \in S$. A more "physical" pictorial interpretation of F is as follows. Draw the set S in the complex plane. To each point z in S attach an arrow with the length and direction of the vector $\mathbf{F}(z)$. We say that such a diagram *represents* the vector field $\mathbf{F}$ (Fig. 10).

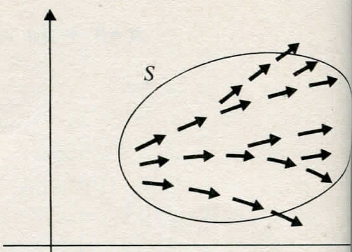


Fig. 10

Conversely, of course, given a vector field $\mathbf{F}$ on S we can define a complex-valued function F of a complex variable, by

$$F(z) = \text{the complex number corresponding to } \mathbf{F}(z).$$

Because of this correspondence between vector fields and complex-valued functions of a complex variable, the two concepts are interchangeable. However, since this course is *complex* analysis, we shall almost always stick to the complex analysis formulation.

In fact, vector fields can also be three-dimensional (but then, of course, we cannot reconstruct a complex-valued function F). They arise in many areas of physics, in time-independent (often called *steady-state*) situations. For example, suppose we have a large river whose flow does not change from day to day. Then we can associate with each point in the river the *velocity vector* of particles passing that point. The function which maps the position of each point to the velocity at that point is a *velocity field*. Given a static system of electric charges, a small stationary "test charge" at a point experiences a force (of attraction or repulsion in a certain direction), which is a vector depending on the position of the point: thus we can construct a vector field, which we would call a *force field*. A more familiar but simpler example is the force due to gravity experienced by objects close to the earth's surface. Many vector fields in physics are velocity or force fields, although the "velocity" or "force" may be rather less "material" than in these examples (for example, later on we shall consider heat flow).

As well as complex-valued functions we can also have real-valued functions of a complex variable—in this context these are often referred to as **scalar fields**. There is often a close connection between scalar and vector fields, and since scalar fields seem simpler, we shall investigate them first. The next definition is crucial.

Definition

Let ϕ be a real-valued function defined, with continuous partial derivatives, on a region S. We define the **gradient** of ϕ, written grad ϕ or $\nabla\phi$, by

$$\nabla\phi(z) = \frac{\partial\phi}{\partial x}(z) + i\frac{\partial\phi}{\partial y}(z), \quad z \in S.$$

Example 3

Let $\phi(z) = \log|z|$, $z \neq 0$. Calculate $\nabla\phi$.

Solution

We have already shown, in the solution to Problem 1 of Section 16.3, that $\dfrac{\partial\phi}{\partial x}(z) = \dfrac{x}{x^2+y^2}$, $\dfrac{\partial\phi}{\partial y}(z) = \dfrac{y}{x^2+y^2}$, where $z = x + iy$. Thus,

$$\nabla\phi(z) = \frac{x+iy}{x^2+y^2} = \frac{1}{x-iy} = \frac{1}{\bar{z}}.$$

If you have come across $\nabla\phi$ before, you will probably recall that it is normally thought of as a vector field; however, using the isomorphism J (page 25), our version and the usual one are equivalent.

The complex-valued function $\nabla\phi$ "codes" all the information about the rate of change of ϕ with respect to position in the complex plane. To make this clear, we shall first show that the vector corresponding to $\nabla\phi(z)$ has the direction of the *largest* directional derivative of ϕ at z. Recall from Section 16.1 that the directional derivative $\nabla_\theta\phi(z)$ of ϕ at z in the direction θ is equal to

$$\frac{\partial\phi}{\partial x}(z)\cos\theta + \frac{\partial\phi}{\partial y}(z)\sin\theta;$$

but by the definition of $\nabla\phi$ and inner product (given in the Corollary to Theorem 7), this is just

$$\nabla\phi(z)\bullet(\cos\theta, \sin\theta).$$

Now by the original definition of inner product, this is greatest when the angle between the two vectors is zero, that is

$$\theta \text{ is the direction of } \nabla\phi(z);$$

and then $\nabla_\theta\phi(z) = |\nabla\phi(z)|$.

We may think of $\phi(z)$ as the height of a hillside above the point z. Then $\nabla\phi(z)$ points in the direction of steepest ascent since $\nabla\phi(z)$ measures the direction of quickest increase of ϕ at z; accordingly, we would expect $\nabla\phi(z)$ to be *normal* to the level curve of ϕ through z (that is, perpendicular to the tangent). In order to make this precise, we need another definition.

Definition

Let γ be a differentiable arc whose domain contains the real number t, such that $\gamma'(t) \neq 0$.

(1) The **tangent vector to** γ **at** $\gamma(t)$ is the vector $\mathbf{t}$ corresponding to $\gamma'(t)$.

(2) A vector $\mathbf{v}$ is **normal** to γ at $\gamma(t)$ if $\mathbf{v}$ is perpendicular to the tangent vector $\mathbf{t}$, that is $\mathbf{v}\bullet\mathbf{t} = 0$.

(3) The **unit normal** to γ at $\gamma(t)$ is the vector $\mathbf{n} = (-i)\mathbf{t}/|\mathbf{t}|$.

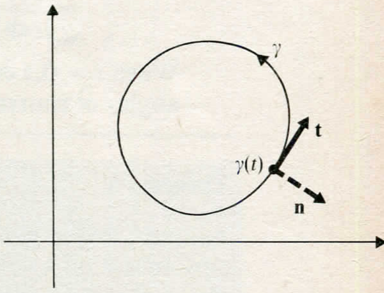


Fig. 11

There are two possible definitions of the unit normal which would make it a vector of length 1 perpendicular to the tangent. We have chosen the one which makes the unit normal point outwards if the path of γ is a simple-closed contour traversed in the positive direction (that is, anticlockwise). See Fig. 11.

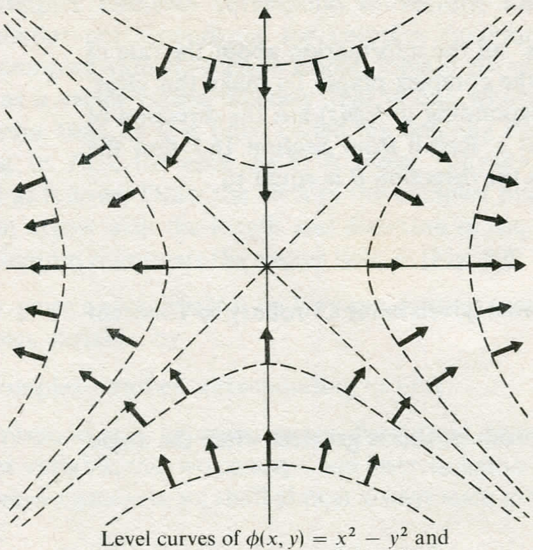
(The physical interpretation of the tangent vector should be known to avid readers of *Unit M231 5, Differentiation*: if $z = \gamma(t)$ is the law giving the position z of a moving particle at the time t, then $\gamma'(t)$ represents the *velocity vector* of the particle at time t.)

Self-Assessment Questions

2. If $\phi(x, y) = x^2 - y^2$, sketch the level curves of ϕ, find grad ϕ and sketch the corresponding vector field on the same graph as the level curves.
3. If $\gamma(t) = \cosh t + i\sinh t$, calculate the tangent vector and unit normal when $t = 0$.

Solutions

2.



Level curves of $\phi(x, y) = x^2 - y^2$ and corresponding vector field. (N.B. Lengths of arrows are not to scale.)

Fig. 12

$$\operatorname{grad} \phi(z) = 2x + i(-2y) = 2(x - iy).$$

3. Tangent vector **t** corresponds to $\gamma'(t) = \sinh t + i \cosh t$. Unit normal vector **n** corresponds to

$$\frac{\cosh t - i \sinh t}{\sqrt{\cosh^2 t + \sinh^2 t}}.$$

When $t = 0$, **t** corresponds to i, and **n** corresponds to 1; **n** and **t** are at right angles of course.

Theorem 8

Suppose that ϕ is a real-valued function having partial derivatives which are continuous on a neighbourhood of a point α, and is such that $\nabla\phi(\alpha) \neq 0$. Then the level curve Γ of ϕ at height $\phi(\alpha)$ has a tangent vector, and $\nabla\phi(\alpha)$ is normal to Γ at α.

Proof

Since $\nabla\phi(\alpha) \neq 0$, either $\dfrac{\partial\phi}{\partial x}(\alpha) \neq 0$ or $\dfrac{\partial\phi}{\partial y}(\alpha) \neq 0$.

Suppose that $\dfrac{\partial\phi}{\partial y}(\alpha) \neq 0$ (the other case is similar). Then by the Implicit Function Theorem there is a real function f differentiable at $a = \operatorname{Re} \alpha$ such that for (x, y) in some neighbourhood of α, $f(x) = y \Leftrightarrow \phi(x + iy) = \phi(\alpha)$. Let $\gamma(t) = (t, f(t))$: then γ has path Γ on a neighbourhood of α, and

$$\gamma'(\alpha) = (1, f'(a)) = \left(1, -\frac{\partial\phi}{\partial x}(\alpha) \bigg/ \frac{\partial\phi}{\partial y}(\alpha)\right) \neq 0.$$

Hence Γ has a tangent vector at α, and also

$$\nabla\phi(\alpha) \bullet \gamma'(\alpha) = \frac{\partial\phi}{\partial x}(\alpha) \cdot 1 - \frac{\partial\phi}{\partial y}(\alpha) \frac{\partial\phi}{\partial x}(\alpha) \bigg/ \frac{\partial\phi}{\partial y}(\alpha) = 0.$$

Hence $\nabla\phi(\alpha)$ is perpendicular to $\gamma'(\alpha)$, that is, normal to Γ at α. ■

There is an important corollary of this result which applies to harmonic functions.

Corollary

Let U be harmonic on a neighbourhood of a point α, with harmonic conjugate V, and suppose that $\nabla U(\alpha)$ and $\nabla V(\alpha)$ are nonzero. Then the tangents at α to the curves $U(z) = U(\alpha)$ and $V(z) = V(\alpha)$ are perpendicular.

Proof

By the Cauchy–Riemann equations,

$$\frac{\partial U}{\partial x}(\alpha) = \frac{\partial V}{\partial y}(\alpha) \quad \text{and} \quad \frac{\partial U}{\partial y}(\alpha) = -\frac{\partial V}{\partial x}(\alpha);$$

so that

$$\nabla U(\alpha) \bullet \nabla V(\alpha) = \frac{\partial U}{\partial x}(\alpha)\frac{\partial V}{\partial x}(\alpha) + \frac{\partial U}{\partial y}(\alpha)\frac{\partial V}{\partial y}(\alpha) = 0,$$

that is $\nabla U(\alpha)$ and $\nabla V(\alpha)$ are perpendicular.

By applying Theorem 8 to U and V, it follows that the tangents are also perpendicular (Fig. 13). ■

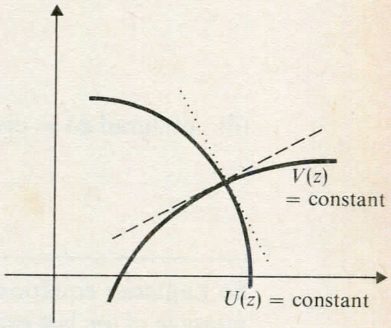


Fig. 13

The "operator" grad constructs a vector field from a scalar field; there are two other operators acting in the opposite direction, which also have great physical significance.

Definition

Let F be a complex-valued function defined on a region S. Let $U = \operatorname{Re} F$ and $V = \operatorname{Im} F$. Suppose that U and V have continuous partial derivatives on S. We define two real-valued functions: the **divergence** of F, written div F, and **circulation** of F, written curl F, by

$$\operatorname{div} F = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y},$$

$$\operatorname{curl} F = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}.$$

Example 4

Calculate div F and curl F when (i) $F(z) = \exp z$, (ii) $F(z) = \bar{z}$.

Solution

(i) We have $U(x, y) = e^x \cos y$, $V(x, y) = e^x \sin y$, so that

$$[\operatorname{div} F](x, y) = e^x(\cos y + \cos y) = 2e^x \cos y,$$

$$[\operatorname{curl} F](x, y) = e^x(\sin y - (-\sin y)) = 2e^x \sin y.$$

(ii) We have $U(x, y) = x$, $V(x, y) = -y$, so that

$$[\operatorname{div} F](x, y) = 1 + (-1) = 0,$$

$$[\operatorname{curl} F](x, y) = 0 - 0 = 0.$$

We can compose the "operators" grad, div and curl provided that we are careful about the order, and that the appropriate second partial derivatives exist.

Example 5

Calculate (i) curl (grad ϕ), (ii) div (grad ϕ).

Solution

(i) $$\operatorname{curl}(\operatorname{grad}\phi) = \operatorname{curl}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)$$

$$= \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}$$

$= 0$, by the Commuting Theorem of Section 16.1.

(ii) $$\operatorname{div}(\operatorname{grad}\phi) = \operatorname{div}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \nabla^2\phi.$$

So Laplace's equation is coming into view again. In this light let us extract the message of the last example. If F corresponds to a vector field and is of the form grad ϕ for some harmonic function ϕ, then curl $F = 0$ and div $F = 0$. It turns out that the conditions curl $F = 0$ and div $F = 0$ frequently crop up in physical situations. It is therefore worthwhile to ask: If curl F = div $F = 0$, is F of the form grad ϕ for some harmonic function ϕ? (If you have studied mechanics then you will have met this question before.)

The answer is yes, for simply-connected regions, and this fact is usually proved by using standard methods of vector analysis, in particular Green's Theorem (for something on this see the Appendix to *Unit M231 11*, *Techniques of Integration*). However, since this course is Complex Analysis, we shall prove it using complex variable techniques.

Now that we want to get back to complex analysis, it is natural to look at the effect of div and curl on analytic functions, since an analytic function gives a vector field. We might expect that div $F = 0$ if F is analytic, but Example 4(i) with F = exp shows that this is not so. Example 4(ii) is more informative, though: in this case the *conjugate* of an analytic function has zero divergence.

Let us look at the conditions div $F = 0$ and curl $F = 0$. They say

$$\frac{\partial U}{\partial x} = -\frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = +\frac{\partial V}{\partial x}.$$

These are the Cauchy–Riemann equations except for a minus sign in front of V: they say just that $U - iV$, that is $\bar{F}$, is analytic.

Thus the following result is now clear.

Theorem 9

Let $F = U + iV$, where U and V have continuous partial derivatives on a region S. Then

div $F = 0$ and curl $F = 0$ on S if and only if $\bar{F}$ is analytic on S.

Using this result, we can answer the above question.

Theorem 10 (The Harmonic Potential Theorem)

Let $F = U + iV$, where U and V have continuous partial derivatives on a simply-connected region S. Suppose that div F = curl $F = 0$ on S. Then there is a function ϕ harmonic on S such that $F = -\nabla\phi$ on S.

Proof

By Theorem 9, $\bar{F}$ is analytic on S and so $-\bar{F}$ is analytic on S. By the Antiderivative Theorem, there is G analytic on S such that $G' = -\bar{F}$ (this is where we require S to be simply-connected). Let $G = \phi + i\psi$. Then ϕ is harmonic. Also

$$U - iV = \bar{F} = -G' = -\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right)$$

$$= -\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}, \quad \text{by the Cauchy–Riemann equations,}$$

and so

$$F = U + iV = -\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}\right) = -\nabla\phi. \quad \blacksquare$$

You might think that it would be mathematically more natural to have $F = +\nabla\phi$: this is probably true, but the minus sign comes in from physics.*

The proof of this theorem is rather similar to the proof of Theorem 2 on harmonic conjugates, and a similar corollary can be drawn if S is not simply-connected.

There is some more terminology that goes with the Harmonic Potential Theorem.

Definition

If F satisfies the conditions of the Harmonic Potential Theorem and $F = -\nabla\phi$, then ϕ is called a **potential function** for F; the level curves of ϕ are called **equipotentials**; the harmonic conjugate ψ of ϕ is called a **stream function**; the level curves of ψ are called **streamlines**, and $G = \phi + i\psi$ is called a **complex potential**.

In some cases G is quite useful; it is well worth noting that $\nabla\phi = \overline{G'}$. We shall see the physical significance of these concepts in the next reading section. There is no need to commit these terms to memory now; you will become familiar with them through usage.

Self-Assessment Questions

4. (i) Calculate div F and curl F when $F(z) = z^2$.

   (ii) Why is ∇F where $F(z) = z^2$, not defined?

5. Show that $\bar{F}: z \longrightarrow \overline{\exp z}$ is not analytic by using Theorem 9 and Example 4.

Solutions

4. (i) We have $U(x, y) = x^2 - y^2$, $V(x, y) = 2xy$; so

$$[\text{div } F](x, y) = 2x + 2x = 4x$$

and

$$[\text{curl } F](x, y) = 2y - (-2y) = 4y.$$

   (ii) F is not real-valued.

---

* For example, if ϕ measures temperature, and F measures heat flow, then, since we regard heat as flowing "downhill" from hot to cold, using the laws of heat we derive an equation of the form $F = -k\nabla\phi$ (where k is a positive constant). This equation expresses the fact that heat flows in the direction of the greatest *decrease* in temperature. Also the conditions $F = \nabla\phi_1$ for some ϕ_1 and $F = -\nabla\phi_2$ for some ϕ_2, are mathematically equivalent (take $\phi_2 = -\phi_1$), and so we can be guided by physics.

5. If $\bar{F}: z \longrightarrow \overline{\exp z}$ were analytic, then by Theorem 9 F must be such that $\operatorname{div} F = 0$ and $\operatorname{curl} F = 0$. But from Example 4, we know this is not the case, so $\bar{F}$ is not analytic.

To conclude we interpret Cauchy's Theorem in terms of vector analysis. As before, the conjugation operation will be important. Let Γ be the path of a smooth arc $\gamma: [a, b] \longrightarrow \mathbf{C}$, and f be continuous on Γ. Then $\int_\Gamma f(z)\, dz$ is defined to be $\int_a^b f(\gamma(t))\gamma'(t)\, dt$. Since the complex number $\gamma'(t)$ corresponds to the tangent vector to Γ at $\gamma(t)$, we can interpret this integral in terms of our operations on vectors, by using Theorem 7. It is convenient to write F for $\bar{f}$: then, by Theorem 7,

$$\bar{F}(\gamma(t))\gamma'(t) = F(\gamma(t)) \bullet \gamma'(t) + i[F(\gamma(t)) \wedge \gamma'(t)],$$

so that

$$\int_\Gamma \bar{F}(z)\, dz = \int_a^b [F(\gamma(t)) \bullet \gamma'(t)]\, dt + i\int_a^b [F(\gamma(t)) \wedge \gamma'(t)]\, dt.$$

There is a standard terminology for the two real integrals on the right-hand side. We define

$$\int_\Gamma F(z) \bullet dz \quad \text{to be} \quad \int_a^b [F(\gamma(t)) \bullet \gamma'(t)]\, dt,$$

and

$$\int_\Gamma F(z) \wedge dz \quad \text{to be} \quad \int_a^b [F(\gamma(t)) \wedge \gamma'(t)]\, dt.$$

These two integrals are often called *real line integrals.* You may recall the notation $\int_\Gamma f(z)|dz|$ introduced in Section 5.3 of *Unit 5*; these new ones are very similar, and can be extended to contours in the usual way. Using this new notation, we have

$$\int_\Gamma \bar{F}(z)\, dz = \int_\Gamma F(z) \bullet dz + i\int_\Gamma F(z) \wedge dz.$$

We can now combine the above equality with Theorem 9. Suppose that $F = U + iV$, where U and V have continuous partial derivatives on a simply-connected region S, and that $\operatorname{div} F = \operatorname{curl} F = 0$. Then for any closed contour Γ in S, $\int_\Gamma F(z) \bullet dz = 0$ and $\int_\Gamma F(z) \wedge dz = 0$, by applying Cauchy's Theorem to $\bar{F}$.

Physical Interpretations

Let F represent a vector field $\mathbf{F}$ on a simply-connected region S, let Γ be a contour in S from α to β, and let z and $z + h$ be points on Γ.

First regard F as a force field. Consider a *small* chord $[z, z + h]$. The force on a particle at z in the direction of h is $|F(z)| \cos \theta$, where θ is the angle between $[z, z + h]$ and $F(z)$: see Fig. 14. Thus the work done on a particle in displacing it by a small amount represented by h is approximately $|F(z)| \cos \theta \times |h|$ (by the definition of work), which is $F(z) \bullet h$. The work done in displacing it from α to β along the curve Γ can be found by approximating Γ from $z_0 = \alpha$ to $z_n = \beta$ by a polygonal line $[z_0, z_1, \ldots, z_{n-1}, z_n]$. If all of $|z_{i+1} - z_i|$, $i = 0, 1, \ldots, n - 1$, are small, we can show that the work done in displacing the particle from α to β along Γ is approximately $\sum_{r=1}^{n} F(z_r) \bullet (z_r - z_{r-1})$, and by proceeding much as in the definition of the Riemann integral we can show that the *work* done as the particle moves from α to β along Γ is $\int_\Gamma F(z) \bullet dz$.

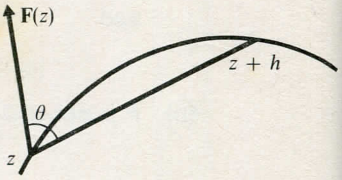


Fig. 14

We could also regard F as the velocity field of some fluid flow. The amount of fluid crossing the small chord $[z, z + h]$ in one unit of time is the velocity perpendicular to $[z, z + h] \times$ length of $[z, z + h]$ which equals $|F(z)| \sin\theta \times |h| = F(z) \wedge h$.

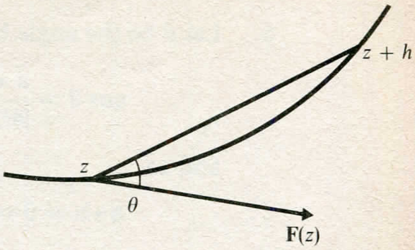


Fig. 15

By the same sort of argument as above, we can show that the total amount of fluid flowing across Γ, or *flux across* Γ, is

$$\int_\Gamma F(z) \wedge dz.$$

Hence,

$$\int_\Gamma \bar{F}(z)\,dz = (\text{work along } \Gamma) + i\,(\text{flux across } \Gamma).$$

This equation should not be interpreted too literally: note, in particular, that the two real integrals come from two *different* interpretations of the vector field F. Nevertheless if curl $F =$ div $F = 0$ (so that $\bar{F}$ is analytic) and Γ is closed, then the work along Γ is zero (if F is interpreted as a force field) and the flux across Γ is zero (if F is interpreted as a velocity field).

Summary

We discussed the correspondence of complex numbers with plane geometric vectors and complex functions with vector fields. We introduced the operators grad, div, and curl and discussed their relevance to analytic functions. The main result is the Harmonic Potential Theorem.

We concluded with a discussion of real line integrals and their relationship to Cauchy's Theorem.

Self-Assessment Questions

6. Show that for nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in G^2, $\mathbf{a} \bullet \mathbf{b} = 0 \Leftrightarrow \mathbf{a}$ is perpendicular to $\mathbf{b}$.
7. Show that $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$.
8. Calculate $a \bullet b$ and $a \wedge b$ when
   (i) $a = 2 + 3i$, $b = 3 + 2i$;
   (ii) $a = 1$, $b = r(\cos\theta + i\sin\theta)$.
9. Calculate $\nabla\phi$ when
   (i) $\phi(x, y) = xe^y$;
   (ii) $\phi(z) = \text{Re}(\text{Log}\, z)$.
10. Calculate the tangent vector to the arc $\gamma(t) = t + it^2$ at the point $1 + i$.
11. Calculate div F and curl F when
$$F(x, y) = x \sin y + i\, ye^x.$$
12. State the Harmonic Potential Theorem.
13. Calculate $\int_C F(z) \bullet dz$ and $\int_C F(z) \wedge dz$, where $F(z) = \dfrac{1}{\bar{z} + i}$ and C is the circle $|z| = 2$.

Solutions

6. Let θ be the angle between **a** and **b**. Then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|},$$

and so

$$\mathbf{a} \bullet \mathbf{b} = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2}$$

$$\Leftrightarrow \mathbf{a} \text{ is perpendicular to } \mathbf{b}.$$

7. In this case $\theta = 0$ and so $\cos\theta = 1$: thus

$$\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|\,|\mathbf{a}|1 = |\mathbf{a}|^2.$$

8. (i) $a \bullet b = 2 \cdot 3 + 3 \cdot 2 = 12;$

$a \wedge b = 2 \cdot 2 - 3 \cdot 3 = -5.$

(ii) $a \bullet b = 1 \cdot r\cos\theta + 0 = r\cos\theta;$

$a \wedge b = 1 \cdot r\sin\theta - 0 = r\sin\theta.$

(See Fig. 16.)

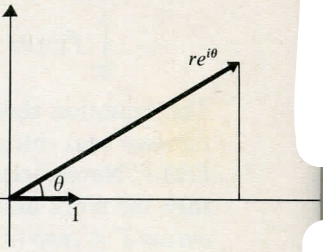


Fig. 16

9. (i) $\nabla\phi(x, y) = e^y + ixe^y = e^y(1 + ix).$

(ii) Let $G(z) = \operatorname{Log} z$. Then G has real part ϕ, and so, by the remark after the definitions (page 33),

$$\nabla\phi(z) = \overline{G'(z)} = \overline{1/z} = 1/\bar{z}.$$

(This is a quicker solution than that in Example 3.)

10. $\gamma'(t) = 1 + 2it$, and so the tangent vector at $1 + i\,(= \gamma(1))$ is the vector corresponding to $\gamma'(1) = 1 + 2i$.

11. $$[\operatorname{div} F](x, y) = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(ye^x)$$

$$= \sin y + e^x.$$

$$[\operatorname{curl} F](x, y) = \frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(x\sin y)$$

$$= ye^x - x\cos y.$$

12. See Theorem 10, page 32.

13. $\bar{F}: z \longrightarrow \dfrac{1}{z - i}$, and so $\displaystyle\int_C \bar{F} = 2\pi i$, from which

$$\int_C F(z) \bullet dz = \operatorname{Re}\int_C \bar{F} = 0$$

and

$$\int_C F(z) \wedge dz = \operatorname{Im}\int_C \bar{F} = 2\pi.$$

16.5 PROBLEMS

1. Let $F(x, y) = \sin x \cosh y - i \cos x \sinh y$.

   (a) Verify that $\operatorname{div} F = \operatorname{curl} F = 0$.

   (b) Hence find a potential function for F.

   (c) What is the stream function in this case?

2. Let ϕ_1 and ϕ_2 be potential functions for F on a region S. Show that $\phi_1 - \phi_2$ is constant on S.

Solutions

1. (a) $[\operatorname{div} F](x, y) = \cos x \cosh y - \cos x \cosh y = 0$,

   and

   $[\operatorname{curl} F](x, y) = -(-\sin x) \sinh y - \sin x \sinh y = 0$.

   (b) By Theorem 9 and part (a), $\bar{F}$ is analytic, where

   $$\begin{aligned}\bar{F}(x, y) &= \sin x \cosh y + i \cos x \sinh y\\ &= \sin(x + iy).\end{aligned}$$

   Also $-\bar{F} = -\sin$ is analytic. Then let $G(z) = \cos z$. Since $G'(z) = -\sin z$, G is a complex potential for F. The potential ϕ is given by

   $$\phi(x + iy) = \operatorname{Re}(\cos(x + iy)) = \cos x \cosh y.$$

   (c) The stream function ψ is given by

   $$\psi(x + iy) = \operatorname{Im}(\cos(x + iy)) = -\sin x \sinh y.$$

2. Since $F = -\nabla\phi_1 = -\nabla\phi_2$, we have

   $$\nabla(\phi_1 - \phi_2) = \nabla\phi_1 - \nabla\phi_2 = 0.$$

   (Note that grad, div and curl are all linear operators, just like partial derivatives.)

   Let $\phi = \phi_1 - \phi_2$; then $\dfrac{\partial\phi}{\partial x} = 0$ and $\dfrac{\partial\phi}{\partial y} = 0$. By the Constancy Condition of Section 16.1, ϕ is constant on S.

16.6 THE PHYSICAL BACKGROUND TO HARMONIC FUNCTIONS

The treatment in this section is necessarily somewhat sketchy, but even if you do not know any physics, it should be enough to give you some feeling for the potential problems of the next section. Do not spend too long poring over this section if you find it hard.

Laplace's equation crops up in many different areas of physics, for example fluid flow, electrostatics, magnetostatics, and gravitation. Here we shall discuss two areas of physics giving rise to Laplace's equation, which are perhaps rather closer to daily experience than some others we might have chosen. These are the flow of fluids, and heat flow.

Fluid Flow

What we present here is a *mathematical model* of the way that water in a river flows. Like all such models, it is not totally accurate, but the approximate answers it gives are worthwhile.

We make certain assumptions about the fluid and the flow. We assume the fluid is *non-viscous*, that is, there is no internal friction between bits of the fluid, and also that there is no friction, or *drag*, as it is often called, between the fluid and any boundary walls. We also assume that the fluid is *incompressible*, that is, completely resistant to change in volume. A non-viscous incompressible fluid is called an *ideal fluid*. (This model also works well for gases which are moving at *low* speeds.)

Here are our basic assumptions about the flow. We suppose that the fluid is flowing over part of a plane (which we take to be the complex plane or z-plane), and that the depth is constant, say one unit. We assume that the velocity at a point does not depend on the depth of the point (but may well depend on its position in the z-plane); in other words, we can regard the flow as *two-dimensional*, and concentrate only on the points on the z-plane. We also assume that the velocity at each point is independent of time, that is, the flow is *steady*.

More particular assumptions are that there are no *sources* or *sinks*, that is, points in the flow at which fluid is entering or leaving, and that the fluid is not rotating in any way—in particular there must be no *vortices*, that is, points around which the fluid is whirling or eddying.

With all these assumptions, you might think that such flows were unrealistic, but in fact, they do provide useful models of some actually occurring flows, for example, the large-scale slow flow of water in a channel and the flow of air round an aircraft wing.

Let S be a simply-connected region of the z-plane in which such a flow is taking place. Let $W(z) = U(z) + iV(z)$ be the velocity vector* at the point z, for each z in S. We shall use the above assumptions to show, in outline, that div $W = 0$.

Consider the small rectangular contour Γ in S, centred on z, with sides of length $2h$ and $2k$ as shown in Fig. 17. The amount Q_Γ of fluid leaving Γ in unit time, that is, the *flux* across Γ, in the sense of Section 16.4, is approximately

$$[U(z+h) - U(z-h)] \cdot 2k \text{ across the sides parallel to the } y\text{-axis}$$

$$+[V(z+k) - V(z-k)] \cdot 2h \text{ across the sides parallel to the } x\text{-axis}$$

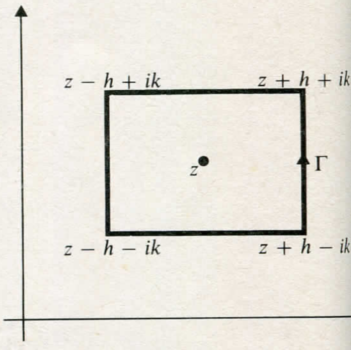


Fig. 17

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* Having established the isomorphism between complex numbers and vectors and having made our definitions (such as tangent vector and unit normal vector), we now find it convenient to write, for example, "Let $W(z) = U(z) + iV(z)$ be the velocity vector . . .".

and so the average loss of fluid per unit area at z is Q_Γ/(area of Γ), that is $Q_\Gamma/4hk$, or, in full,

$$\frac{U(z+h)-U(z-h)}{2h}+\frac{V(z+k)-V(z-k)}{2k}.$$

It is reasonable that this approximation gets better and better as h and k approach zero, so that

$$\begin{aligned}\text{loss at } z &= \lim_{(h,k)\to 0}\left(\frac{U(z+h)-U(z-h)}{2h}+\frac{V(z+k)-V(z-k)}{2k}\right)\\ &= \lim_{h\to 0}\frac{U(z+h)-U(z-h)}{2h}+\lim_{k\to 0}\frac{V(z+k)-V(z-k)}{2k}\\ &= \frac{\partial U}{\partial x}(z)+\frac{\partial U}{\partial y}(z).\end{aligned}$$

But the loss at z is zero because (a) no fluid enters or leaves at z (since z is not a source or sink), and (b) no fluid is stored at z (since the fluid is incompressible). Thus $\dfrac{\partial U}{\partial x}(z)+\dfrac{\partial V}{\partial y}(z)=0$, for all $z \in S$, that is,

$$\operatorname{div} W = 0 \text{ on } S.$$

A similar argument, based on the absence of rotation in S can be used to show that

$$\operatorname{curl} W = 0 \text{ on } S.$$

It follows from the Harmonic Potential Theorem that there is a function ϕ harmonic on S such that $W = -\nabla\phi$. We call ϕ the **velocity potential for the flow** W. The velocity potential itself does not have any particular physical reality or mathematical significance. However, the complex potential G and the stream function ψ (using the notation on page 33) are much more informative.

Note first that $\nabla\phi = \overline{G'}$, and so the velocity vector $W(x, y)$ can be calculated by

$$W(x, y) = -\overline{G'(x+iy)};$$

so that in particular the *speed* $|W(x, y)|$ of the flow W at a point (x, y) is $|G'(x + iy)|$.

Secondly, note that the velocity $W(x, y)$ is normal to the curve $\phi =$ constant through (x, y) and so (by the corollary to Theorem 8) it is tangential to the streamline $\psi =$ constant through (x, y). We can interpret this to mean that the fluid particles tend to move along the streamlines, and if we can determine the streamlines we have a good overall picture of the flow.

It remains only to interpret the boundary conditions. Let Γ be the path of a smooth arc which is part of the boundary of S, and z_0 be a point on Γ (Fig. 18). Since Γ is impervious, there can be no flow of fluid in a direction normal to Γ, and so the velocity vector $W(z_0)$ is tangential to Γ; thus Γ is a streamline.

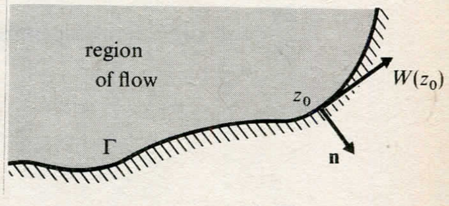


Fig. 18

There is an alternative formulation of this condition which is more convenient. Let **n** be the unit normal to Γ at z_0: then $W(z_0) \bullet \mathbf{n} = 0$, that is $\nabla\phi(z_0) \bullet \mathbf{n} = 0$. Now $\nabla\phi(z_0) \bullet \mathbf{n}$ is the directional derivative of ϕ at z_0 in the direction of **n**, and is often called the **normal derivative** at z_0, and abbreviated to $\dfrac{\partial\phi}{\partial n}(z_0)$.

Thus Γ is a streamline if and only if $\dfrac{\partial\phi}{\partial n} = 0$ on Γ. Hence, the standard boundary condition for fluid flow is that $\dfrac{\partial\phi}{\partial n} = 0$ on the boundary of S. (Note, however, that we have to exclude from consideration those points of Γ at which **n** is not defined. Such a point would be a sharp corner of Γ.)

Let us take the most basic example of fluid flow. Suppose water is flowing across all of the z-plane with uniform velocity, at speed α in a direction parallel to the x-axis. Then $W(x, y) = \alpha$, for all x and y. We can determine G by the equation $W(x, y) = -\overline{G'(x + iy)}$, so that

$$G'(x + iy) = \overline{-\alpha} = -\alpha,$$

and so $G(z) = -\alpha z + \beta$, for some constant β; we shall take $\beta = 0$.

Thus the stream function ψ is given by

$$\psi(x, y) = \operatorname{Im}(G(x + iy)) = -\alpha y,$$

and so the streamlines are the lines $-\alpha y = k$, for constants k, that is, lines parallel to the x-axis (Fig. 19). This is just what one would expect.

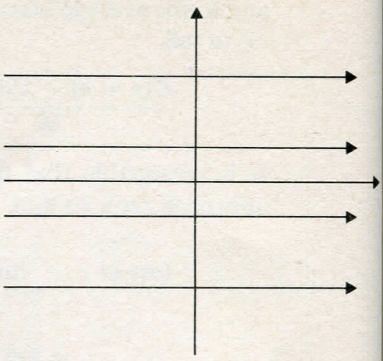


Fig. 19

Heat Flow

Consider a thin plate S, insulated on both faces (thus allowing no flow of heat across them) and maintained at various prescribed temperatures on the edges, with perhaps certain edges insulated. Fig. 20 shows one such case, in which one edge is insulated, and the three others are maintained at temperatures of 0, 50 and 100.

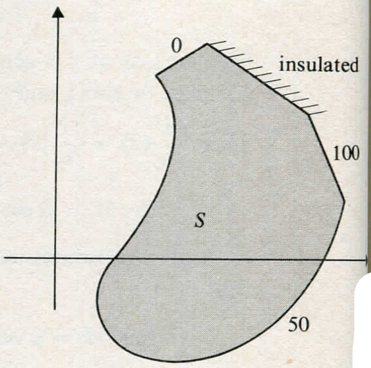


Fig. 20

After some lapse of time, the temperature $T(x, y)$ at any fixed point (x, y) in the plate will be constant. It can be shown from the physical laws of heat that this temperature function T is harmonic, that is $\nabla^2 T = 0$, on S. In this case the temperature is the *observed* quantity. (Recall that for the fluid flow the counterpart of T, namely ϕ, is not an observed quantity.) We can calculate the heat flow from the physical law

$$W = -k\nabla T,$$

where k is a positive real number, called the *conductivity*, which depends on the material of the plate. If H is a harmonic conjugate of T, then the analytic function $T + iH$ is the complex potential. The curves of constant temperature $T(x, y) =$ constant are called *isotherms* (you may have seen them in an atlas). The curves $H(x, y) =$ constant give the streamlines of heat flow.

The boundary conditions for heat flow are like those in Fig. 20. If Γ is the path of a smooth arc on the boundary of S, we can have either (a) $T =$ constant on Γ (corresponding to Γ being maintained at a prescribed temperature) or (b) $\dfrac{\partial T}{\partial n} = 0$ on Γ (corresponding to Γ being insulated).

Summary

Our main topic was the two-dimensional steady flow of a non-viscous incompressible fluid with no sources, sinks, or rotation. We outlined a proof that the velocity W has $\operatorname{div} W = 0$, and stated that $\operatorname{curl} W = 0$; these facts imply that W has a harmonic potential ϕ. We also discussed the natural boundary condition $\dfrac{\partial \phi}{\partial n} = 0$ for such a flow.

Next we very briefly discussed two-dimensional steady heat flow, mentioning the appropriate boundary conditions.

Self-Assessment Questions

1. Let W be the velocity function for the two-dimensional steady flow of an incompressible non-viscous fluid in a simply-connected region. Under what circumstances will W have a velocity potential?
2. How do you calculate the speed of the flow from the complex potential?
3. What relationship does the stream function have to the flow?
4. What is $\dfrac{\partial \phi}{\partial n}(z)$?
5. What are the usual boundary conditions for fluid flow?
6. What is a potential function for the flow $W(x, y) = \alpha$ in the whole complex plane?
7. What are the usual boundary conditions for heat flow, and what are the physical conditions associated with them?

Solutions

1. When there are no sources (or sinks) or rotation in S, so that div $W =$ curl $W = 0$.
2. If G is the complex potential, then the speed $|W| = |G'|$.
3. The fluid particles tend to move along the streamlines $\psi =$ constant, where ψ is the stream function.
4. If Γ is the path of a smooth arc and z is a point on it, then $\dfrac{\partial \phi}{\partial n}(z)$ is the directional derivative $\nabla_\alpha \phi(z)$ in the direction α of the unit normal to Γ at z.
5. $\dfrac{\partial \phi}{\partial n} = 0$ on the boundary of S. (See page 39.)
6. A complex potential is $G(z) = -\alpha z$, and so a potential function is $\phi(x, y) = -\alpha x$. (See page 40.)
7. See page 40.

16.7 BOUNDARY VALUE PROBLEMS IN POTENTIAL THEORY

The examples in the last section demonstrate that our brief discussion of boundary value problems in Section 16.2 was too restrictive in its choice of conditions for the boundary. We shall formulate a less special version of the Dirichlet problem of Section 16.2, and a new kind of problem, the Neumann problem.

The boundary of the regions under consideration cannot be completely general. From now on we shall consider only regions R with the following property: the boundary B of R is the disjoint union of simple-closed contours or simple generalized contours $\Gamma_1, \ldots, \Gamma_m$, where by "generalized" we mean that they may be unbounded (as opposed to ordinary contours, which are necessarily bounded). Fig. 21 shows such a region.

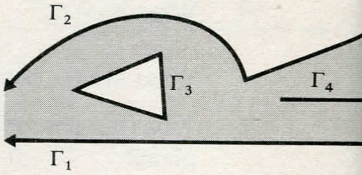


Fig. 21

From now on we shall use "region" in this restricted sense; it will embrace all the topics you are likely to meet.

We also need to restrict the functions under consideration. We say that a function f is **piecewise-continuous** on the boundary B of such a region if f is continuous on B except at finitely many points $p_1, \ldots, p_n$, at which f need not even be defined, provided it is *locally bounded*, that is bounded on a punctured neighbourhood of each p_j. (This is a little more general that the definition used in M201, Linear Mathematics.) A function f is called **piecewise-constant** if each contour Γ_j of the boundary B is the sum of contours on which, apart from the endpoints, f is constant. A piecewise-constant function is piecewise-continuous.

For example, let R be the annulus $\{z : 1 < |z| < 2\}$. Then the function U defined by

$$U(z) = \begin{cases} \operatorname{Re} z, & \text{if } |z| = 1, \\ 1, & \text{if } |z| = 2 \text{ and } \operatorname{Im} z > 0, \\ 0, & \text{if } |z| = 2 \text{ and } \operatorname{Im} z < 0, \end{cases}$$

is piecewise-continuous on the boundary B of R, and is piecewise-constant on the circle $|z| = 2$ (Fig. 22).

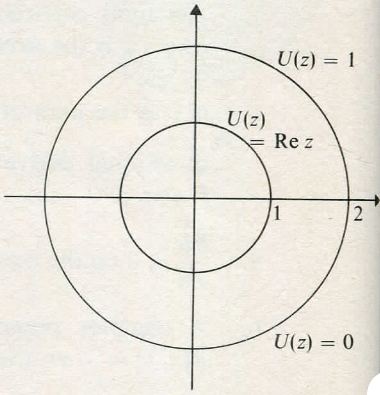


Fig. 22

With these definitions we can formulate two new boundary value problems.

The Dirichlet Problem—a less special case Let R be a region and U_0 be a real-valued function piecewise-continuous on the boundary B of R. Find a function U harmonic on R such that, for all points z on B at which U_0 is continuous, U has limit $U_0(z)$ near z on R.

Note that we cannot now say "U is continuous on $\bar{R}$", as we did in Section 16.2; this is why we have had to formulate the more complicated limit condition above.

(Recall from *Unit 2* that U has limit l near z on R, which we can write as $\lim\limits_{\zeta \to z \text{ in } R} U(\zeta) = l$, if for all $\varepsilon > 0$ there is $\delta > 0$ such that whenever $0 < |\zeta - z| < \delta$ and $\zeta \in R$, then $|U(\zeta) - l| < \varepsilon$.)

The conditions in this problem are not always sufficient to determine U uniquely; if they are not, we usually add some "ancillary conditions" derived from the physics of the problem. These come in two kinds.

(i) If U_0 is not continuous at a point z on B, U should be bounded in $\bar{R}$ on a punctured neighbourhood D of z (that is, U is bounded on $D \cap \bar{R}$). Recall that U_0 is bounded in B on a punctured neighbourhood of z, by the definition of piecewise-continuity.

(ii) If R is unbounded, we should add a "condition at infinity". In fluid flow, where the regions are normally unbounded, it is often reasonable to suppose that far out the flow has constant velocity, that is $\lim\limits_{z \to \infty} W(z) = \alpha$.

The Neumann Problem Let R be a region and $N:B \longrightarrow \mathbf{R}$ a real-valued function continuous on the boundary B of R. Find a function U harmonic on R and continuous on $\bar{R}$ such that, for all points z on B in a neighbourhood of which B is the path of a smooth arc (and so has a unique tangent), $\dfrac{\partial U}{\partial n}(z) = N(z)$.

In practice, the condition $\dfrac{\partial U}{\partial n} = 0$ is most common; this situation is called the *homogeneous form* of the Neumann problem. However, the homogeneous Neumann problem does not usually occur on its own, but as part of a "mixed" Dirichlet–Neumann problem. In fact, it can be shown that if $\dfrac{\partial U}{\partial n} = 0$ on B, and R is simply-connected, then U is constant.

A very useful fact (which we shall not attempt to prove) is that Poisson's Formula (the result of Theorem 6) solves the Dirichlet problem posed here if R is an open disc; in other words, we can relax the assumption "U_0 is continuous" in the hypotheses of Theorem 6 to "U_0 is piecewise-continuous". In practice, U_0 is normally piecewise-constant. Let us, then, see an example of Poisson's Formula in action. (This example is our first problem to be stated in physical terms.)

Example 1

Let D be the open unit disc $|z| < 1$. Let the temperature function U_0 on the boundary C of D have the values 1 in the upper half-plane, and 0 in the lower half-plane. Find the temperature function U at all points of D.

Solution

We observed in the last section (page 40) that U is harmonic. In this case U_0 is piecewise-constant on the boundary C; in fact, $U_0(e^{i\theta}) = 1$ if $0 < \theta < \pi$, and $U_0(e^{i\theta}) = 0$ if $\pi < \theta < 2\pi$, with U_0 undefined at $\theta = 0$ and π. By Poisson's Formula, at the point $z = \rho e^{i\phi}$, where $0 \leqslant \rho < 1$,

$$U(\rho e^{i\phi}) = \frac{1-\rho^2}{2\pi} \int_0^\pi \frac{1}{1+\rho^2 - 2\rho\cos(\theta-\phi)}\, d\theta.$$

To simplify this integral we use the natural substitution $t = \tan\frac{1}{2}(\theta - \phi)$, so that $\theta - \phi = 2\arctan t$, and so

$$d\theta = 2dt/(1+t^2),$$
$$\cos(\theta-\phi) = (1-t^2)/(1+t^2).$$

We obtain the indefinite integral (or antiderivative)

$$\begin{aligned}
&\frac{1-\rho^2}{2\pi}\int \frac{2}{(1+\rho^2)(1+t^2) - 2\rho(1-t^2)}\,dt \\
&= \frac{1}{\pi}\int \frac{(1+\rho)(1-\rho)}{t^2(1+\rho)^2 + (1-\rho)^2}\,dt \\
&= \frac{1}{\pi}\int \frac{1}{1+t^2(1+\rho)^2/(1-\rho)^2}\cdot\frac{1+\rho}{1-\rho}\,dt \\
&= \frac{1}{\pi}\arctan\left(\frac{1+\rho}{1-\rho}t\right);
\end{aligned}$$

and so

$$\begin{aligned}
U(z) &= \frac{1}{\pi}\arctan\left(\frac{1+\rho}{1-\rho}\tan\frac{\theta-\phi}{2}\right)\Bigg|_0^\pi \\
&= \frac{1}{\pi}\left[\arctan\left(\frac{1+\rho}{1-\rho}\tan\frac{\pi-\phi}{2}\right) + \arctan\left(\frac{1+\rho}{1-\rho}\tan\frac{\phi}{2}\right)\right].
\end{aligned}$$

The example serves to show the strength and deficiencies of Poisson's Formula. Using it, we can get a formula for the harmonic function U, and with some ingenuity in substitution, eliminate the integral sign from it. But the resulting formula is not at all transparent: it takes a great deal more ingenious technique to simplify it to

$$U(z) = \frac{1}{\pi}\operatorname{Arg}\left(i \cdot \frac{1+z}{1-z}\right).$$

Thus instead of using Poisson's Formula for the disc, or other regions, in a blunderbuss fashion, we shall apply common sense and a variety of other techniques. (In fact, we shall give another solution to Example 1 later.) In simple cases we can often guess the answer after a little manipulation.

Example 2

Let R be the upper half-plane $\operatorname{Im} z > 0$, and

$$U_0(x) = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x > 0. \end{cases}$$

Find a function U harmonic on R and continuous on $\bar{R} - \{0\}$ such that $U(x) = U_0(x)$ if $x \neq 0$ (where $x \in \mathbf{R}$).

(We could interpret this mathematical formulation as a model of the heat flow in the upper half-plane subject to the given boundary conditions.)

Solution

This is precisely a Dirichlet problem, as formulated above.

Now

$$\operatorname{Arg} x = \begin{cases} \pi, & \text{if } x < 0 \\ 0, & \text{if } x > 0. \end{cases}$$

Also Log is analytic on R and so $\operatorname{Arg} = \operatorname{Im}(\operatorname{Log})$ is harmonic on R. Thus the required function U is given by

$$U(x+iy) = \frac{1}{\pi}\operatorname{Arg}(x+iy) = \frac{1}{\pi}\arccos\frac{x}{\sqrt{x^2+y^2}}.$$

Fig. 23 shows some of the level curves of U.

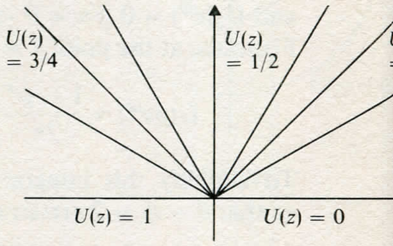


Fig. 23

Note that U is undefined at 0, which is to be expected since there is a discontinuity in U_0 at 0. Note also that the function U we have found satisfies the two "ancillary" boundary conditions mentioned after the Dirichlet problem (page 42): for U is bounded on $D \cap \bar{R}$ for some neighbourhood D of 0 (the condition at the discontinuity 0), and U is bounded on all of R and so, in particular, for large z (the condition at infinity).

It should now be clear that if we specify the boundary condition

$$U_0(x) = \begin{cases} 1, & \text{if } x < p, \\ 0, & \text{if } x > p \end{cases}$$

where p is a point on the real axis, then the harmonic function

$$U(z) = \frac{1}{\pi}\operatorname{Arg}(z-p)$$

satisfies this. By Problem 2 of Section 16.3, any linear combination of harmonic functions is harmonic. Hence, if p and q are real numbers with $p < q$, then the function

$$U(z) = \frac{1}{\pi}\operatorname{Arg}(z-q) - \frac{1}{\pi}\operatorname{Arg}(z-p)$$

is harmonic, and satisfies the boundary condition

$$U_0(x) = \begin{cases} 0, & \text{if } x < p, \\ 1, & \text{if } p < x < q, \\ 0, & \text{if } x > q. \end{cases}$$

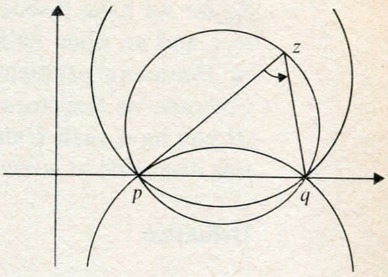


Fig. 24

The geometric interpretation of U is interesting: $U(z) = \frac{1}{\pi} \times$ angle pzq. From this it can be shown that the curves $U(z) =$ constant are circles through p and q (Fig. 24).

Example 3

Let R be the half-plane $\text{Im}\, z > 0$. Let $p_1 < \cdots < p_n$ be points on the real axis. Find a function U harmonic on R satisfying the boundary condition

$$U_0(x) = \begin{cases} \alpha_0, & \text{if } x < p_1 \\ \alpha_k, & \text{if } p_k < x < p_{k+1}, \quad \text{for } k = 1, \ldots, n-1 \\ \alpha_n, & \text{if } p_n < x. \end{cases}$$

Solution

We add together solutions of the previous type.

Clearly

$$U_{k+1}(x + iy) = \frac{\alpha_k}{\pi}[\text{Arg}(z - p_{k+1}) - \text{Arg}(z - p_k)]$$

has boundary values α_k inside (p_k, p_{k+1}) and 0 outside, for $k = 1, \ldots, n-1$. For $k = 0$ or n it is not hard to see how to modify Example 2: we take

$$U_1(x + iy) = \frac{\alpha_0}{\pi} \text{Arg}\,(z - p_1),$$

and

$$U_{n+1}(x + iy) = \frac{\alpha_n}{\pi}[\pi - \text{Arg}(z - p_n)].$$

The required solution is $U_1 + U_2 + \cdots + U_{n+1}$.

There are some other common-sense methods that can be used, but they are best discussed in the context of particular applied potential problems.

Analytic Equivalence

So far we have discussed only solutions of the Dirichlet problem for an open disc and an open half-plane. So what do we do if we are posed the Dirichlet or Neumann problem for another sort of region R? We do something fairly obvious: we transform R into a disc or half-plane by an appropriate function. (If you have read *Unit 13, Properties of Analytic Functions*, you will have seen this discussed at some length.)

Definition

Let R and S be regions. A function f defined on $\bar{R}$ is an **analytic equivalence** of R with S if

(1) f is analytic on R,

(2) f is continuous on $\bar{R}$,

(3) $f(R) = S$ and $f(\bar{R}) = \bar{S}$,

(4) f is one-one on $\bar{R}$,

(5) $f:\bar{R} \longrightarrow \bar{S}$ has an inverse function $g:\bar{S} \longrightarrow \bar{R}$ analytic on S and continuous on $\bar{S}$.

(Some of these properties can be deduced from the others by using theorems from *Unit 13*, but from our standpoint this is unimportant.)

The essence of the definition is that if two regions are **analytically equivalent** (that is, there is an analytic equivalence of one with the other), then they are equivalent for boundary value problems, in a sense that we shall explain shortly.

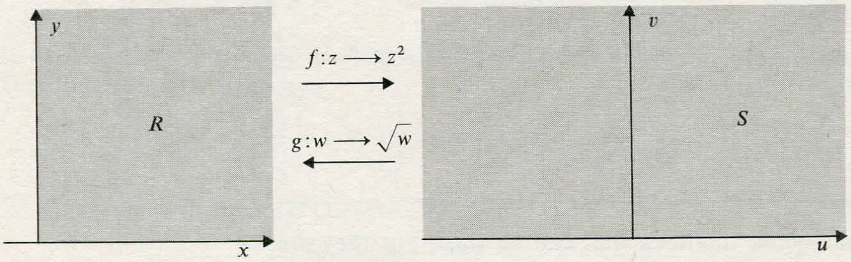


Fig. 25

Fig. 25 shows two regions which are analytically equivalent and the relevant functions f and g. Note that in this context we shall write $z = x + iy$ and $w = u + iv$. (In Section 16.9 we present a table of six analytic equivalences which you will use in this section and in Section 16.8.)

Example 4

Let R be the first quadrant $\{z : \operatorname{Im} z > 0 \text{ and } \operatorname{Re} z > 0\}$.

Find a function U harmonic on R and continuous on $\bar{R} - \{0\}$ satisfying the boundary conditions

$$U_0(x) = 0 \quad \text{if } x > 0,$$

$$U_0(iy) = 1 \quad \text{if } y > 0.$$

(Again this problem could have arisen from a problem of heat flow.)

Solution

Step 1 Let S be the upper half-plane $\{w : \operatorname{Im} w > 0\}$. Define $f : \bar{R} \longrightarrow \bar{S}$ by $f(z) = z^2$. This has an inverse $g : \bar{S} \longrightarrow \bar{R}$ (the principal branch of square root) analytic on S. It is easy to check that f is an analytic equivalence of R with S (Fig. 25). (See entry 1 of the table in Section 16.9 with $n = 2$.)

Step 2 Now $U_0(x + i0) = 0$ and $U_0(0 + iy) = 1$. Define a function U_0^* on the boundary of S by $U_0^*(w) = U_0(g(w))$. Then, writing $w = u + iv$, we have $U_0^*(u + i0) = U_0(g(u + i0)) = 0$ if $u > 0$, and $U_0^*(u + i0) = U_0(g(u + i0)) = 1$ if $u < 0$.

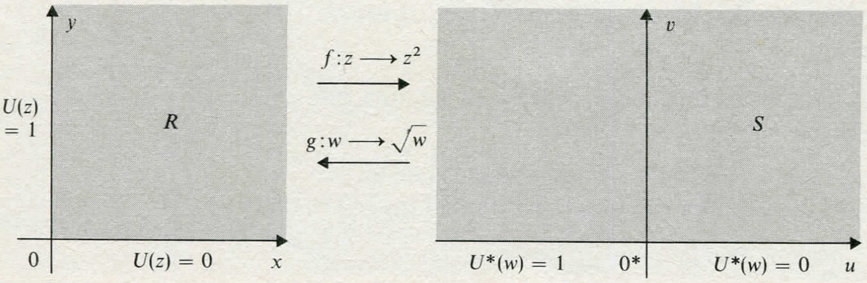


Fig. 26

Fig. 26 illustrates the analytic equivalence and the two sets of boundary conditions.

Step 3 But we have solved this Dirichlet problem (specified in Step 2) for S in Example 2. Let $U^*(u + iv) = \frac{1}{\pi}\operatorname{Arg}(u + iv)$; then U^* is harmonic on S.

Step 4 Now define U by $U(z) = U^*(f(z))$. Thus

$$\begin{aligned} U(x + iy) &= \frac{1}{\pi}\operatorname{Arg}(x + iy)^2 \\ &= \frac{2}{\pi}\operatorname{Arg}(x + iy) \\ &= \frac{2}{\pi}\arccos\frac{x}{\sqrt{x^2 + y^2}}. \end{aligned}$$

This method seems very quick, but it has left several questions unanswered, such as "Is U harmonic?". So let us analyse the strategy in more detail.

Problem (A Dirichlet Problem)

Let R be a region and B be the boundary of R; let $U_0 : B \longrightarrow \mathbf{R}$ be continuous on B. Find a function U harmonic on R, continuous on $\bar{R}$ such that $U = U_0$ on B.

Algorithm for Solution

Step 1 Find an analytic equivalence f of R with S. (In our case we shall do this by looking up the table of analytic equivalences in Section 16.9.)

Step 2 Define a function U_0^* on the boundary B^* of S by

$$U_0^*(w) = U_0(g(w)), \quad w \in B^*,$$

(where g is the inverse function of f). Since $w \in \bar{S} - S$, we have $g(w) \in g(\bar{S}) - g(S)$, and so $g(w) \in \bar{R} - R$; in other words, $U_0(g(w))$ is defined. Hence

$$U_0^* : B^* \longrightarrow \mathbf{R},$$

and U_0^* is continuous, since it is the composition of two continuous functions. (You might like to construct a diagram showing the functions f, g, U, U^*, U_0 and U_0^*.)

Step 3 By solving the Dirichlet problem for U_0^* on S, find a function U^* harmonic on S and continuous on $\bar{S}$ such that $U^*(w) = U_0^*(w)$, $w \in B^*$.

Step 4 Define U on $\bar{R}$ by $U(z) = U^*(f(z))$, $z \in \bar{R}$. Then U is harmonic on R (by Problem 3 of Section 16.3). Also U is continuous on $\bar{R}$ (since U^* and f are continuous). Finally, if $z \in \bar{R} - R$, we have $f(z) \in \bar{S} - S$, and so

$$U(z) = U^*(f(z)) = U_0^*(f(z)) = U_0(g(f(z))) = U_0(z),$$

since g is the inverse of f. Thus U is a solution to the Dirichlet problem for U_0 on R.

Steps 2 and 4 above are automatic: it is steps 1 and 3 which may give trouble, and in fact there is no general guarantee that they can be carried out. However, very often in practice they can.

The same method works if U_0 is piecewise-continuous (and so justifies our solution of Example 4)—see Self-Assessment Question 5.

Using an analytic equivalence of the unit disc with the upper half-plane we can use the algorithm above and give a much easier solution to Example 1 (page 43).

Second Solution to Example 1

Step 1 Let $f(z) = \frac{1}{i}\cdot\frac{z-1}{z+1}$, D be the open disc $|z| < 1$, and S be the upper half-plane. Then f is an analytic equivalence of D with S. (See entry 2 of the table in Section 16.9.) Now

$$U_0(z) = \begin{cases} 1, & \text{if } |z| = 1 \text{ and } \operatorname{Im} z > 0, \\ 0, & \text{if } |z| = 1 \text{ and } \operatorname{Im} z < 0. \end{cases}$$

Step 2 Let $U_0^*(w) = U_0(g(w))$, where $g = f^{-1}$. Then

$$U_0^*(w) = \begin{cases} 1, & \text{if } w \in \mathbf{R} \text{ and } w > 0, \\ 0, & \text{if } w \in \mathbf{R} \text{ and } w < 0. \end{cases}$$

Step 3 Hence, we can take $U^*(w) = 1 - \frac{1}{\pi}\operatorname{Arg} w$ (by Example 3).

Step 4 Finally, let $U(z) = U^*(f(z))$; then

$$\begin{aligned} U(z) &= \frac{1}{\pi}(\pi - \operatorname{Arg} f(z)) \\ &= \frac{1}{\pi}\left[\operatorname{Arg}(-1) - \operatorname{Arg}\left(\frac{1}{i}\cdot\frac{z-1}{z+1}\right)\right]. \end{aligned}$$

Now $\operatorname{Arg}(-1) - \operatorname{Arg} w = \operatorname{Arg}(-1/w)$, as you can check if you want to, and so

$$U(z) = \frac{1}{\pi}\operatorname{Arg}\left(\frac{-i(z+1)}{z-1}\right) = \frac{1}{\pi}\operatorname{Arg}\left(i\cdot\frac{1+z}{1-z}\right).$$

This is the answer suggested after the first solution to Example 1,

Recall that U is the temperature in Example 1. To get a good idea of the temperature distribution it is very useful to draw the isotherms. Note that $0 \leqslant U(z) \leqslant 1$ for $z \in D$, and so the isotherms are the curves $U(z) = k$, where $0 \leqslant k \leqslant 1$. Now

$$\begin{aligned} U(z) = k &\Leftrightarrow \frac{1}{\pi}\left[\pi - \operatorname{Arg}\left(\frac{1}{i}\cdot\frac{z-1}{z+1}\right)\right] = k \\ &\Leftrightarrow \operatorname{Arg}\left(\frac{1}{i}\cdot\frac{z-1}{z+1}\right) = \pi - \pi k \\ &\Leftrightarrow \operatorname{Arg}(z-1) - \operatorname{Arg}(z+1) = \pi - \pi k + \frac{\pi}{2} (\operatorname{mod} 2\pi), \text{ since } \operatorname{Arg} i = \frac{\pi}{2}. \end{aligned}$$

Thus the isotherms are the circles

$$\operatorname{Arg}(z-1)-\operatorname{Arg}(z+1)=c, \quad \text{where } -\frac{\pi}{2}\leqslant c\leqslant\frac{\pi}{2},$$

through 1 and -1 (Fig. 27). (See the discussion above Example 3.)

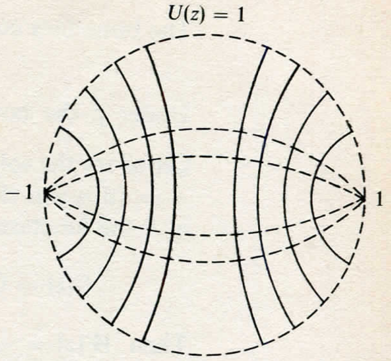


Fig. 27

Finally, homogeneous Neumann boundary conditions can be incorporated: this fact depends on the following lemma, which we shall not prove.

Lemma

Let f be an analytic equivalence of R with S. Let Γ be a contour forming part of the boundary of R, and let Γ^* be its image under f. If U^* is a function harmonic on S which satisfies $\dfrac{\partial U^*}{\partial n}=0$ on Γ^*, then the function $U=U^*\circ f$, which is harmonic on R, satisfies $\dfrac{\partial U}{\partial n}=0$ on Γ.

In order to use these methods, we have to find analytic equivalences. The basic result on these is a famous theorem of Riemann. (If you have read *Unit 13*, you will have seen it before. In fact, the version we state is weaker than that given in *Unit 13*.)

The Riemann Mapping Theorem

Let R be a simply-connected region which is the inside of a simple-closed contour. Then there is an analytic equivalence of R with the open unit disc. Also there is an analytic equivalence of R with the open upper half-plane.

This result, while important theoretically, does not give us any practical way of *finding* analytic equivalences. As we have said, we shall not expect you to find them unaided: you are expected to use the table in Section 16.9.

Analytic equivalences are especially useful in fluid flow. Consider the following example.

Example 5

A fluid flows across the z-plane except for a circular obstacle of radius 1 with centre the origin. The velocity far from the obstacle is $\alpha(>0)$.

(a) Find the complex potential G for the flow.

(b) Sketch some of the streamlines.

Solution

(*a*) Since there is symmetry about the x-axis, we need consider only the upper half-plane outside the obstacle. This gives us a simply-connected region R. From entry 3 of the table in Section 16.9, there is an analytic equivalence f of R with S, the upper half-plane, given by $f(z)=z+\dfrac{1}{z}$. Fig. 28 shows this equivalence and corresponding arcs and points.

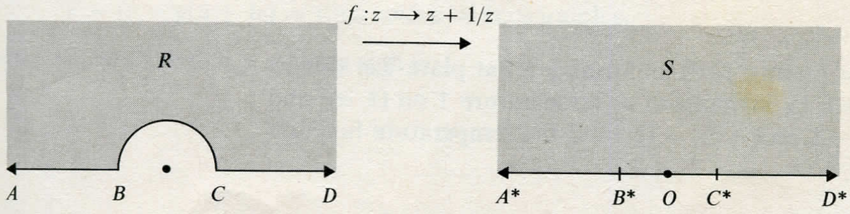


Fig. 28

The boundary condition for $G = \phi + i\psi$ is $\dfrac{\partial \phi}{\partial n} = 0$ on $\bar{R} - R$.

Under f, the boundary condition transforms to $\dfrac{\partial \phi^*}{\partial n} = 0$ on the u-axis. Let us try the solution for the complex potential G^* in S given by $G^*(w) = -\alpha w$ (this was discussed in Section 16.6). This gives as complex potential in R the function

$$G(z) = G^*(f(z)) = -\alpha\left(z + \frac{1}{z}\right).$$

Then $W(z) = \overline{-G'(z)} = \alpha(1 - \overline{1/z^2})$, and so the velocity far from the obstacle (that is, for large z) is α, as required by the "condition at infinity". Hence G is the required complex potential. (Note that we have used analytic equivalences to find complex potentials. The justification for this is straightforward.)

(b) We can now calculate the potential ϕ and stream function ψ in R, since $\phi(z) = \operatorname{Re} G(z)$ and $\psi(z) = \operatorname{Im} G(z)$. It is useful to set $z = re^{i\theta}$. Then

$$\begin{aligned}\phi(re^{i\theta}) &= -\alpha \operatorname{Re}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right)\\ &= -\alpha\left(r + \frac{1}{r}\right)\cos\theta,\end{aligned}$$

and

$$\psi(re^{i\theta}) = -\alpha\left(r - \frac{1}{r}\right)\sin\theta.$$

The streamlines are the curves $-\alpha\left(r - \dfrac{1}{r}\right)\sin\theta = k$. Fig. 29 shows some of them—they look much as you would expect; however, note that the circle $|z| = 1$ and the real axis form a streamline (the one where $k = 0$).

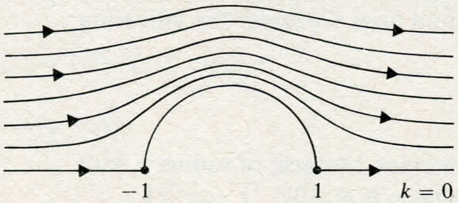


Fig. 29

We define z_0 to be a **stagnation point** of a flow with velocity field W if $W(z_0) = 0$. In the above example, z_0 is a stagnation point if

$$\alpha\left(1 - \overline{\frac{1}{z_0^2}}\right) = 0, \text{ that is } \overline{z_0^2} = 1,$$

so that the stagnation points are 1 and -1.

Our final example on the method of analytic equivalence is taken from heat flow and illustrates a "mixed" Dirichlet–Neumann boundary-value problem.

Example 6

Consider the half-plane $R = \{z : \operatorname{Im} z > 0\}$ as forming a flat plate. Let this be insulated on the interval $(-1, 1)$, maintained at temperature 1 on $(1, \infty)$ and maintained at temperature -1 on $(-\infty, -1)$. Find the temperature function T on R.

Solution

Let S be the semi-infinite strip $\{z:|\mathrm{Re}\, z| < \pi/2, \mathrm{Im}\, z > 0\}$.

Then S is analytically equivalent to R, under the function $g:\bar{S} \longrightarrow \bar{R}$ given by $g(w) = \sin w$—see entry 4 of the table in Section 16.9. (Note that S is a fundamental region for g and so g has an analytic inverse $f:\bar{R} \longrightarrow \bar{S}$.) The boundary conditions on R for T transform on S to

$$T^*(-\pi/2 + iv) = -1, \quad T^*(\pi/2 + iv) = 1, \quad \text{for } v > 0,$$

and

$$\frac{\partial T^*}{\partial n} = 0 \text{ on } (-\pi/2, \pi/2).$$

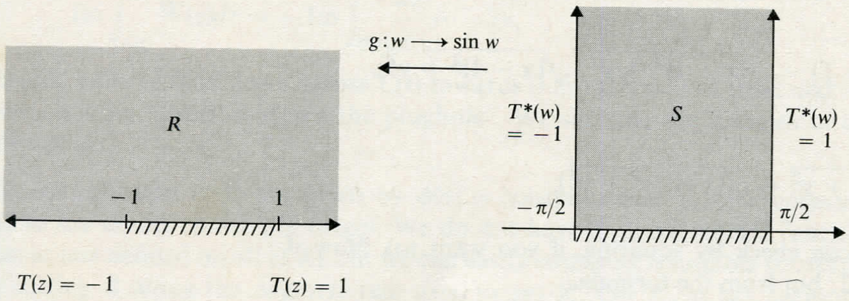


Fig. 30

We have to find T^*. Now heat flows from the side on which $T^*(w) = 1$ to the side on which $T^*(w) = -1$. We now proceed by inspired trial and error. By the symmetry of the situation, this flow seems likely to be parallel to the u-axis (in the negative direction). By analogy with fluid flow, let us try a linear temperature of the form

$$T^*(u, v) = \alpha u + \beta, \text{ for certain } \alpha, \beta \in \mathbf{R}.$$

Then

$$1 = T^*(\pi/2, v) = \alpha(\pi/2) + \beta,$$

and

$$-1 = T^*(-\pi/2, v) = -\alpha(\pi/2) + \beta,$$

so that $\beta = 0$ and $\alpha = 2/\pi$.

Hence, $T^*(u, v) = 2u/\pi$ satisfies the Dirichlet boundary conditions. Also, since $\dfrac{\partial T^*}{\partial n} = \dfrac{\partial T^*}{\partial v} = 0$ on $(-\pi/2, \pi/2)$, it satisfies the Neumann condition too.

Hence, the required temperature T on R is given by

$$T(x, y) = \frac{2}{\pi}\mathrm{Re}\big(f(x + iy)\big),$$

which we now have to simplify.*

Let $u + iv = f(x + iy)$. Then

$$x + iy = g(u + iv) = \sin u \cosh v + i \cos u \sinh v,$$

and so $x = \sin u \cosh v$, $y = \cos u \sinh v$.

---

* We would not expect you to be able to perform unaided the simplification which follows.

Thus

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = \cosh^2 v - \sinh^2 v = 1,$$

so that

$$x^2(1 - \sin^2 u) - y^2 \sin^2 u = \sin^2 u(1 - \sin^2 u),$$

and so

$$(\sin^2 u)^2 - (x^2 + y^2 + 1)\sin^2 u + x^2 = 0.$$

Thus

$$\sin^2 u = \tfrac{1}{2}[(x^2 + y^2 + 1) - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}],$$

using the formula for roots of a quadratic. (We have taken the minus sign in front of the root because $\sin^2 u \leqslant 1$.) We now use the formula for the difference of two squares, to obtain

$$\sin^2 u = \tfrac{1}{2}[(x^2 + y^2 + 1) - \sqrt{(x + 1)^2 + y^2} \cdot \sqrt{(x - 1)^2 + y^2}],$$

and so

$$\sin u = \frac{\sigma}{2}[\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2}],$$

where $\sigma = 1$ or -1 (as you can check by squaring, if you want to). Now if $x > 1$ and $y = 0$, then $u = \pi/2$; but from the formula,

$$\sin u = \frac{\sigma}{2}[\sqrt{(x + 1)^2} - \sqrt{(x - 1)^2}],$$

so that $\sin \pi/2 = \sigma$, and so $\sigma = 1$.

Hence,

$$\begin{aligned} T(x, y) &= \frac{2}{\pi} u \\ &= \frac{2}{\pi} \arcsin(\tfrac{1}{2}[\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2}]). \end{aligned}$$

More concisely, $T(x, y) = \dfrac{2}{\pi} \arcsin \tfrac{1}{2}(r_2 - r_1)$, where r_1 and r_2 are the distances shown in Fig. 31.

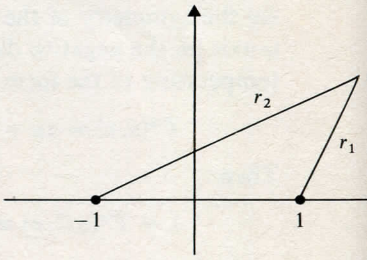


Fig. 31

Singularities of Potential

Our final topic is potential problems in regions which are *not* simply-connected. A standard example of such a region is the punctured plane $\mathbf{C} - \{0\}$, and so let us begin by considering the physical significance of the harmonic function $\phi(z) = \log|z|$ discussed in Problem 1 of Section 16.3.

Example 7

Suppose that $\phi(z) = \log|z|$ is the velocity potential for a fluid flow. What is the velocity? Describe the flow in general terms.

Solution

Let $R = \mathbf{C} - \{0\}$. Then ϕ is harmonic on R. For any point z in R there is a complex potential L analytic on a neighbourhood of z, and such that $\phi = \operatorname{Re} L$; in fact, L is a branch of the logarithm. Thus the velocity at z is given by

$$W(z) = -\overline{L'(z)} = -1/\bar{z}.$$

(Alternatively, $W(z) = -\nabla\phi(z) = -1/\bar{z}$, from Example 3 of Section 16.4.)

Hence, if z lies on the circle $C(r)$ of radius r and centre 0, then

$$|W(z)| = \frac{1}{r};$$

in fact if $z = re^{i\theta}$, then

$$W(z) = \frac{-1}{re^{-i\theta}} = -\frac{1}{r}e^{i\theta}.$$

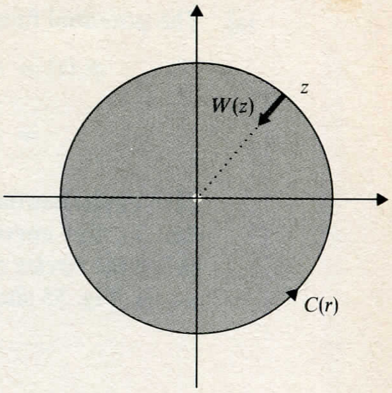


Fig. 32

Thus at every point on the circle $C(r)$ the speed of the flow is $1/r$ and the direction is towards the origin (Fig. 32). The fluid is rushing towards the origin in a radially symmetric way and vanishing there. We call the origin a **sink** of the flow. We can find out how much is vanishing by calculating the flux across $C(r)$. From the formula on page 35, this is

$$\operatorname{Im}\int_{C(r)} \overline{W}(z)dz = -\operatorname{Im}\int_{C(r)} \frac{dz}{z} = -2\pi.$$

Thus 2π units of fluid flows across $C(r)$ towards 0 in each unit of time, and the same quantity "vanishes down the plughole". We say that the *strength* of the sink 0 is -1.

The equipotential lines are given by $\phi(z) =$ constant, that is, $|z| =$ constant, and so are circles centre the origin. We do not have a global stream function (that is, one defined on all of R), but we can easily construct a suitable substitute by cutting R along the negative real axis, to get R'. On R', ϕ has harmonic conjugate $\psi = \operatorname{Arg}$, so that the streamlines are the curves $\operatorname{Arg} z =$ constant, that is, straight lines through the origin (Fig. 33). Clearly the negative real axis is also a streamline, so that the cut did not affect the flow.

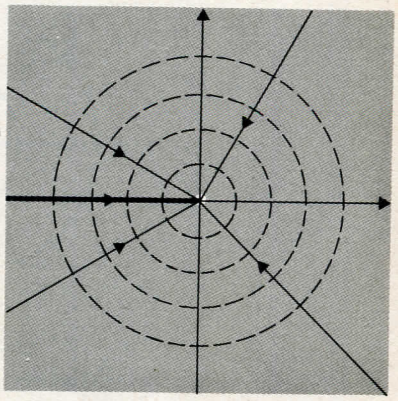


Fig. 33

In general, if a flow W has velocity potential $\phi(z) = -k \log |z|$, then we say that the origin 0 is a **singularity of strength** k. If k is positive, then we say that 0 is a **source**; if k is negative, then we say that 0 is a **sink**. The same terminology applies to the point z_0 if the velocity potential is $\phi(z) = -k \log |z - z_0|$. Note that near a source, the potential is large and positive, whereas near a sink, the potential is large and negative.

There is another kind of flow connected with singularities. The function $\phi = \operatorname{Arg}$ is harmonic on the cut plane R', and so ϕ is the velocity potential for a flow. The conjugate function to Arg is $\psi : z \longrightarrow -\log |z|$, and since this gives the streamlines of the flow and is defined on $R = \mathbf{C} - \{0\}$, we can regard the flow as existing on R, with the "many-valued" potential function arg. The streamlines of this flow are circles centre the origin, traversed in the positive (that is, anticlockwise) sense (Fig. 34).

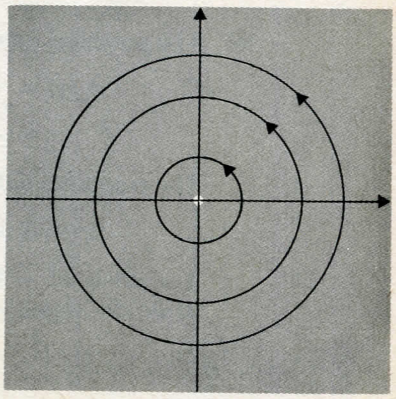


Fig. 34

We call the origin a **vortex**, and the flow a *vortex flow*. The analogy with whirlpools (in bath water, for example) is obvious.

Sources and sinks also occur in heat flow: a source is a point at which heat enters, and a sink a point at which it leaves.

Example 8

Sketch the equipotential lines and the streamlines of the flows given by

(a) a source of strength 1 at 1 and a sink of strength -1 at -1;

(b) a source of strength 1 at 1 and a source of strength 1 at -1.

Solution

Let R be the region $\{z \in \mathbf{C} : z \neq -1, 1\}$. The potential of a source (or sink) of strength α at z_0 is $\phi(z) = -\alpha \log |z - z_0|$. Since ∇^2 is a linear operator, the potential is the sum of the two separate potentials. (See Problem 2 of Section 16.3.)

(a) The potential function is

$$\begin{aligned}\phi_1(z) &= -\log|z-1| + \log|z+1| \\ &= -\log\left|\frac{z-1}{z+1}\right|.\end{aligned}$$

The equipotentials are the curves $-\log|z-1|/|z+1| = k$, for $k \in \mathbf{R}$, that is, the curves $|z-1| = c|z+1|$ for real $c > 0$. These are circles, called the *circles of Apollonius*, except in the case $c = 1$, when we obtain the y-axis. Fig. 35 shows the equipotential lines and the streamlines.

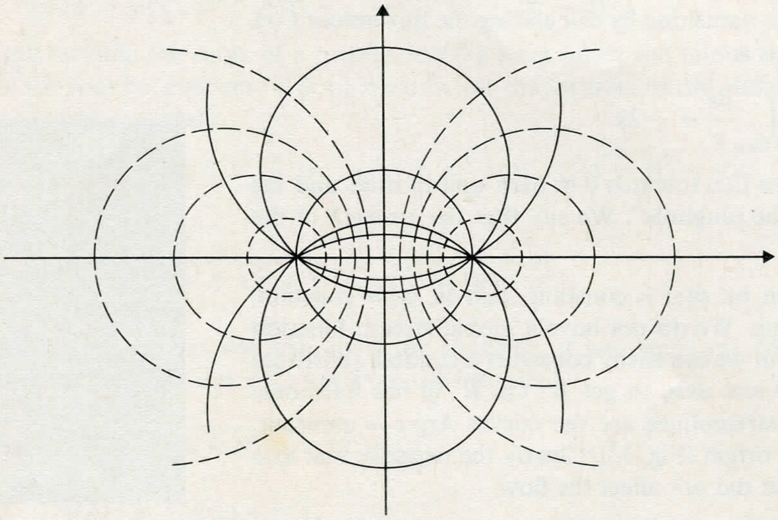


Fig. 35

(b) The potential function is

$$\begin{aligned}\phi_2(z) &= -\log|z-1| - \log|z+1| \\ &= -\log|z^2-1|.\end{aligned}$$

The equipotentials for ϕ_2 are the curves

$$|z^2 - 1| = c, \quad \text{for real } c > 0.$$

The curve $|z^2 - 1| = 1$ is called a *lemniscate*, and the others are called the *ovals of Cassini*. Fig. 36 shows the equipotential lines and the streamlines.

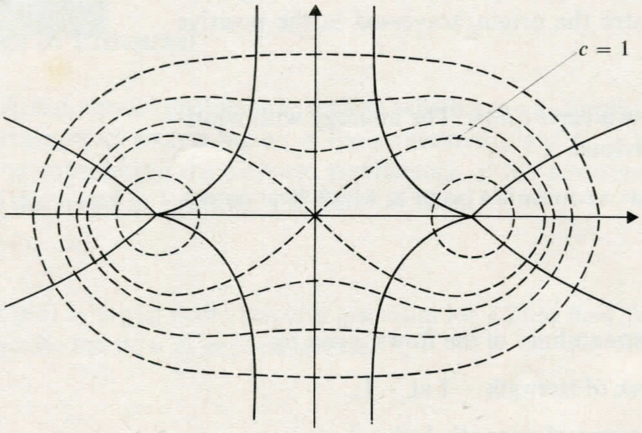


Fig. 36

The following example is typical of a method which we can apply to many situations.

Example 9

Consider $R = \{z: \operatorname{Re} z > 0\}$. Let there be a fluid source of strength 1 at 1. Find the potential function for the flow.

Solution

We observe that in Example 8(b) the y-axis is a streamline and so $\dfrac{\partial \phi}{\partial n} = 0$ on the y-axis. Since this boundary condition is satisfied, the potential function $\phi(z) = -\log|z^2 - 1|$, $\operatorname{Re} z \geqslant 0$, gives the solution to Example 9.

To summarize: the flow in R due to the one fluid source of strength 1 at 1, with the y-axis as boundary, is the same, on R, as that generated by a source at 1, and a source of the same strength at -1. This second source is located at the *image* of 1 under reflection in the y-axis. However, since this source is not physically present in the region R of actual flow, it is usually called a **virtual source**.

A number of problems can be solved in a manner similar to Example 9: the technique is to add virtual sources (or sinks) in a symmetric way so that the boundaries of the region of flow become streamlines. This technique is called the **method of images**. Once we have found a solution by such a method of trial and error, we know that it must be *the* solution, by using uniqueness theorems.

Summary

This is the most important section in the unit, where all the previous insights and techniques culminate. We began by discussing the Dirichlet and Neumann problems. After quickly going over the use of Poisson's Formula for a disc to solve Dirichlet problems, we discussed a technique for solving the Dirichlet problem on the upper half-plane (Examples 2 and 3).

The really powerful technique is the method of analytic equivalence. Using this, we can transform a Dirichlet or Neumann problem on a region to a similar problem on the unit disc or upper half-plane. We then discussed several applications of this.

Finally we considered the problem of singularities of potential. We looked at sources, sinks and (briefly at) vortices, and ended by considering pairs of sources (or sinks). By considering one of the pair as a virtual source (sink) we were able to solve more problems—this technique was an example of the Method of Images.

Self-Assessment Questions

1. State the Dirichlet problem.
2. State the Neumann problem.
3. What is the relevance of Poisson's Formula to the Dirichlet problem?
4. Suppose that R is the upper half-plane, and U_0 is a function piecewise-constant on the x-axis. Describe how you would solve this Dirichlet problem.
5. Describe in general terms the algorithm for solving the Dirichlet problem for a piecewise-continuous function U_0 and region R analytically equivalent to the open unit disc S.
6. What are sources and sinks?
7. What is a vortex?
8. What is a stagnation point?

Solutions

1. See page 42.
2. See page 43.
3. Poisson's Formula solves the Dirichlet problem for an open disc; see Example 1 and the paragraph preceding it.
4. This is just the problem in Example 3. Read that together with Example 2.
5. *Step 1* Find an analytic equivalence f of R with S.

   Step 2 Define U_0^* on $\bar{S} - S$ by $U_0^*(w) = U_0(g(w))$, $w \in \bar{R} - R$. This will be piecewise-continuous on $\bar{S} - S$.

   Step 3 By Poisson's Formula or otherwise, find a function U^* harmonic on S such that $\lim\limits_{\eta \to w \text{ in } S} U^*(\eta) = U_0^*(w)$, for all $w \in \bar{S} - S$ at which U_0^* is continuous.

   Step 4 Define U on R by $U(z) = U^*(f(z))$, $z \in R$. Then U is harmonic on R. Also (since continuous functions preserve limits), $\lim\limits_{\zeta \to z \text{ in } R} U(\zeta) = U_0(z)$, for all $z \in \bar{R} - R$ at which U_0 is continuous.
6. See page 53.
7. See page 53.
8. See page 50.

16.8 PROBLEMS

1. (a) Find the complex potential, and hence the velocity function W, for the fluid flow in the quadrant $R = \{z:\operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$. (You may assume that the velocity for $\operatorname{Re} z$ large is approximately α, where $\alpha > 0$. (Hint: Use $z \longrightarrow z^2$.)

   (b) Find the stagnation points of the flow.

   (c) Sketch some streamlines.

2. (a) Let S be the upper half-plane $\{z:\operatorname{Im} z > 0\}$. Suppose that the temperature on the real axis is given by

$$U(x) = \begin{cases} 0, & \text{if } x < -1 \\ 1, & \text{if } -1 < x < 0 \\ 0, & \text{if } x > 0. \end{cases}$$

   Calculate the temperature distribution U in S. (See Example 2 and paragraph following it in Section 16.7.)

   (b) Let R be the semi-infinite sector $\{z:0 < \operatorname{Arg} z < \pi/4\}$. Suppose that the side $\operatorname{Arg} z = 0$ is held at temperature 0, and side $\operatorname{Arg} z = \pi/4$ at temperature 1 if $|z| < 1$ and 0 if $|z| > 1$. Calculate the temperature function T in R. (Hint: Make use of the result of part (a).)

3. (a) What is the potential function ϕ_h induced by a source of strength Q at the point $h\ (>0)$ and a sink of strength $-Q$ at 0?

   (b) Let $Q = 1/h$. Calculate ϕ where

$$\phi(z) = \lim_{h \to 0^+} \phi_h(z).$$

   (You may assume that $\displaystyle\lim_{h \to 0^+} \left(1 - \frac{h}{z}\right)^{1/h} = e^{-1/z}$.)

   (c) Show that ϕ is harmonic, on a suitable region.

   (d) Calculate the associated stream function, and sketch some streamlines.

4. (a) Let R be the quadrant $\{z:\operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$. A source of strength 1 lies at the point $1 + i$. By considering an analytic equivalence of R with the upper half-plane, and then using a virtual source, calculate the resulting potential function ϕ for the fluid flow, assuming that the velocity at infinity is zero.

   (b) Now calculate the potential function by using three virtual sources outside R. Do the two methods give the same answer?

   (c) Calculate the velocity function. Are there any stagnation points?

   (d) Sketch some streamlines.

5. (a) Let R be the upper half of the unit disc. Find the temperature function T in R if $T = 0$ on the negative real axis, $T = 1$ on the positive real axis, and the semi-circular part is insulated.

   (b) Sketch some isotherms and streamlines.

Solutions

1. (a) We map R to the upper half-plane S using the analytic equivalence $f: z \longrightarrow z^2$ (Fig. 37).

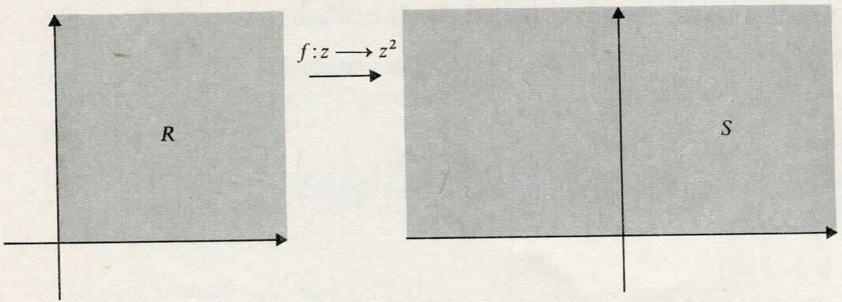


Fig. 37

The complex potential in S is $G^*(w) = -\alpha w$ (see Example 5 of Section 16.7). Hence, the complex potential in R is

$$G(z) = G^*(f(z)) = -\alpha z^2.$$

Thus the velocity function W is given by

$$W(z) = -\overline{G'(z)} = 2\alpha\bar{z};$$

in other words, $W(x, y) = 2\alpha x - i2\alpha y$.

(b) The only stagnation point is the origin.

(c) The stream function ψ is given by $\psi(z) = \operatorname{Im} G(z)$, so that

$$\psi(x, y) = -\alpha \operatorname{Im}(x + iy)^2 = -2\alpha xy.$$

Thus, the streamlines are the curves $\psi(x, y) =$ constant, that is

$$xy = k, \quad k \geqslant 0.$$

These are hyperbolas, for $k > 0$, and the axes, for $k = 0$ (Fig. 38).

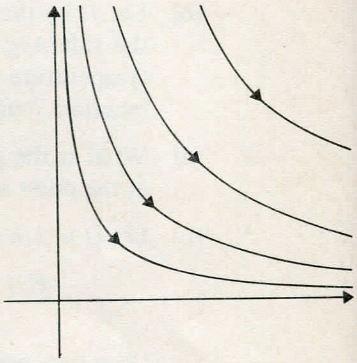


Fig. 38

2. (a) By the result of the discussion after Example 2, U is given by

$$U(z) = \frac{1}{\pi}(\operatorname{Arg} z - \operatorname{Arg}(z + 1)).$$

(If you have read *Unit 14*, you should compare this problem with Example 3 of Section 14.1.)

(b) We map R to the upper half-plane S by the analytic equivalence $f: z \longrightarrow z^4$.

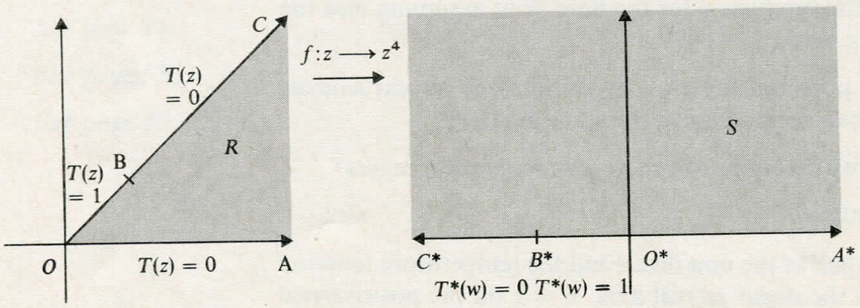


Fig. 39

The boundary conditions transform to the following (note that $f(e^{i\pi/4}) = -1$):

$$T^*(u + i0) = \begin{cases} 0, & \text{if } u > 0 \\ 1, & \text{if } -1 < u < 0 \\ 0, & \text{if } u < -1. \end{cases}$$

Hence, we have transformed the original problem to that of part (a), and so we identify the functions U and T^*. Thus T is defined by

$$\begin{aligned} T(z) &= T^*(z^4) \\ &= \frac{1}{\pi}(\operatorname{Arg} z^4 - \operatorname{Arg}(z^4 + 1)) \\ &= \frac{1}{\pi}(4 \operatorname{Arg} z - \operatorname{Arg}(z^4 + 1)). \end{aligned}$$

(We could express $T(x + iy)$ in terms of x and y, but it would be a little tedious to do directly. The isotherms, $T =$ constant, could be obtained by using a computer.)

3. (a) The potential of a source (or sink) of strength α at z_0 is $z \longrightarrow -\alpha \log|z - z_0|$. (See page 53.) Hence,

$$\begin{aligned}\phi_h(z) &= -Q\log|z-h| - (-Q)\log|z-0| \\ &= -Q\log\left|\frac{z-h}{z}\right|.\end{aligned}$$

(b) Since $Q = 1/h$,

$$\phi_h(z) = -\log\left|\frac{z-h}{z}\right|^{1/h} = -\log\left|\left(1-\frac{h}{z}\right)^{1/h}\right|.$$

Thus

$$\begin{aligned}\phi(z) &= -\lim_{h\to 0^+}\log\left|\left(1-\frac{h}{z}\right)^{1/h}\right| \\ &= -\log\left|\lim_{h\to 0^+}\left(1-\frac{h}{z}\right)^{1/h}\right| \\ &= -\log|e^{-1/z}|, \quad \text{by the result given,} \\ &= -\log(1/|e^{1/z}|) \\ &= \log|e^{1/z}|. \\ &= \operatorname{Re}(1/z), \quad \text{since } |e^{1/z}| = e^{\operatorname{Re}(1/z)}.\end{aligned}$$

(c) The function $\phi(z) = \operatorname{Re}(1/z)$ is harmonic on $\mathbf{C} - \{0\}$.

(d) $\phi(z) = \operatorname{Re}(1/z)$ has harmonic conjugate $\psi(z) = \operatorname{Im}(1/z)$, that is

$$\psi(x, y) = -\frac{y}{x^2 + y^2}.$$

This is the associated stream function.

The streamlines are the curves $\psi(x, y) =$ constant, that is,

$$x^2 + y^2 + cy = 0, \quad \text{for } c \in \mathbf{R}.$$

These are circles with centres on the y-axis and tangential to the x-axis (Fig. 40). The potential function $\phi(z) = k\operatorname{Re}(1/z)$ is said to be due to a *dipole* at 0 of *strength* k. (In the context of magnetism, you may realize that a dipole is equivalent (in mathematical terms) to a small bar magnet. Fig. 40 may well remind you of the alignment of iron filings caused by the magnetic field of a small bar magnet.)

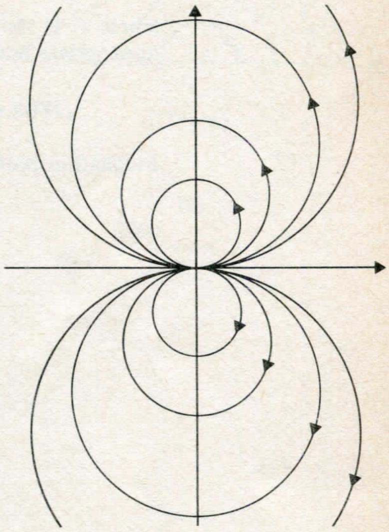


Fig. 40

4. (a) Let S be the upper half-plane, and consider the analytic equivalence $f: z \longrightarrow z^2$ of R with S (Fig. 41).

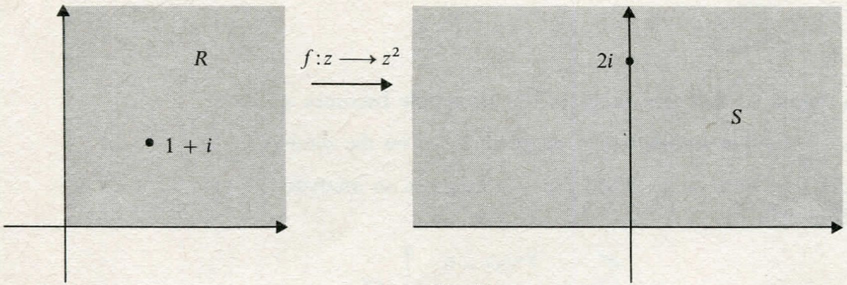


Fig. 41

Now $(1 + i)^2 = 2i$. The potential due to a source of strength 1 at $2i$ in S is given by

$$\begin{aligned}\phi^*(w) &= -\log|w - 2i| - \log|w + 2i|, \\ &\qquad \text{(using a virtual source at } -2i \text{ so that } y = 0 \text{ is a streamline),} \\ &= -\log|w^2 - (2i)^2| \\ &= -\log|w^2 + 4|.\end{aligned}$$

Hence, $\phi(z) = \phi^*(f(z)) = -\log|z^4 + 4|$.

(b) Since $y = 0$ is a streamline, the source at $1 + i$ must be matched by a virtual source at $1 - i$. Similarly, since $x = 0$ is a streamline, the source at $1 + i$ must be matched by a virtual source at $-1 + i$. Also, since 0 is a stagnation point, we should have a third virtual source at $-1 - i$. (See Fig. 42.)

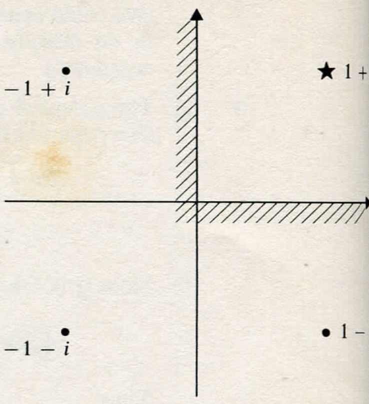


Fig. 42

The potential due to this system is

$$\begin{aligned}\phi(z) &= -\log|z - (1 + i)| - \log|z - (1 - i)| \\ &\quad -\log|z - (-1 + i)| - \log|z - (-1 - i)| \\ &= -\log|(z - 1)^2 + 1| - \log|(z + 1)^2 + 1| \\ &= -\log|(z^2 + 2 - 2z)(z^2 + 2 + 2z)| \\ &= -\log|(z^2 + 2)^2 - 4z^2| \\ &= -\log|z^4 + 4|.\end{aligned}$$

Thus the two methods give the same answer. (However, method (a) is more general, since it will work for any semi-infinite sector, of angle $\pi\alpha$, where $0 < \alpha < 1$, by using the analytic equivalence $z \longrightarrow z^{1/\alpha}$, whereas method (b) will work only for rational α, since these are the only values for which it is possible to get an equivalent finite system of virtual sources.)

(c) The velocity W is given by

$$W(z) = -\overline{G'(z)},$$

where G is the local complex potential at z. Since $G(z) = -L(z^4 + 4)$ for an appropriate branch L of the logarithm,

$$W(z) = \frac{4\bar{z}^3}{\bar{z}^4 + 4}.$$

Stagnation points occur where $W(z) = 0$. Thus the only stagnation point is 0.

(d)

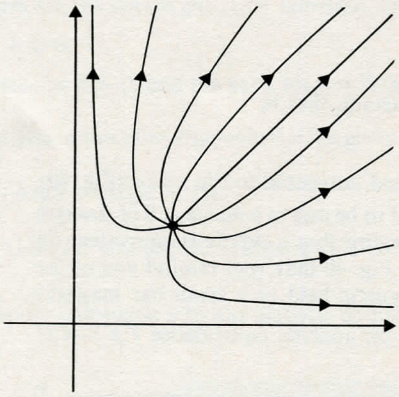


Fig. 43

5. (a) We must map R onto a region so that the image of the semicircle becomes a streamline, because then $\dfrac{\partial T^*}{\partial n} = 0$ is automatically satisfied. Let S be the semi-infinite strip $\{w : \operatorname{Re} w < 0, 0 < \operatorname{Im} w < \pi\}$. Then $f : z \longrightarrow \operatorname{Log} z$ is an analytic equivalence of R with S (Fig. 44).

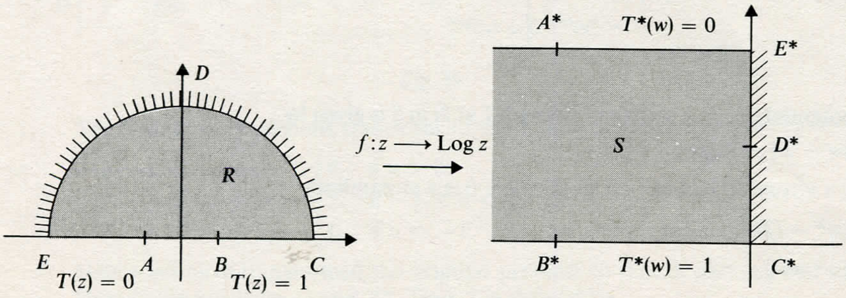


Fig. 44

The transformed boundary conditions are

$$T^*(w) = 0 \text{ on the line } v = \pi, \quad T^*(w) = 1 \text{ on the line } v = 0,$$

and

$$\frac{\partial T^*}{\partial n}(w) = 0 \text{ on } [0, \pi i].$$

By inspection (or recall Example 6 of Section 16.7), $T^*(u, v) = 1 - v/\pi$ is a solution in S, so that $T^*(w) = 1 - \frac{1}{\pi}\,\mathrm{Im}\, w$. Hence,

$$\begin{aligned} T(z) &= T^*(f(z)) \\ &= 1 - \frac{1}{\pi}\,\mathrm{Im}(\mathrm{Log}\, z) \\ &= 1 - \frac{1}{\pi}\,\mathrm{Arg}\, z. \end{aligned}$$

Thus, $T(x, y) = 1 - \frac{1}{\pi}\arctan\frac{y}{x}$.

(If you are skilled enough, you might have done this by guesswork!)

(b) The isotherms are given by $T(z) =$ constant, that is, $\mathrm{Arg}\, z = k$, for $0 \leqslant k \leqslant \pi$. In other words, they are straight lines through the origin. It should be almost clear from this that the streamlines are circles: we can verify this by observing that the harmonic conjugate of T is

$$H(z) = 1 + \frac{1}{\pi}\log|z|.$$

Thus the curves $H(z) =$ constant are just circles $|z| = k$, where $0 < k < 1$. Fig. 45 shows the isotherms and the streamlines.

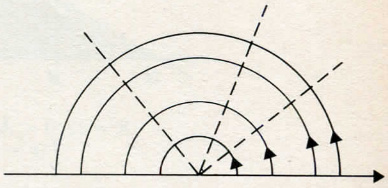


Fig. 45

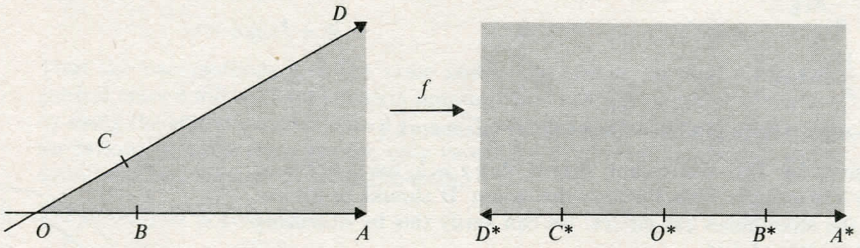
16.9 A TABLE OF ANALYTIC EQUIVALENCES

As we have said before the purpose of this table is to enable you to select an appropriate analytic equivalence rather than having to *construct* it.

In the figures various points and their images are marked. Such correspondences are denoted, for example, by B and B^* in the first entry.

Since the "regions" we consider may have "generalized" contours as part of their boundaries, we need to be able to denote "points at infinity". This is achieved by means of an arrow; see, for example A^* and D^* in the first entry.

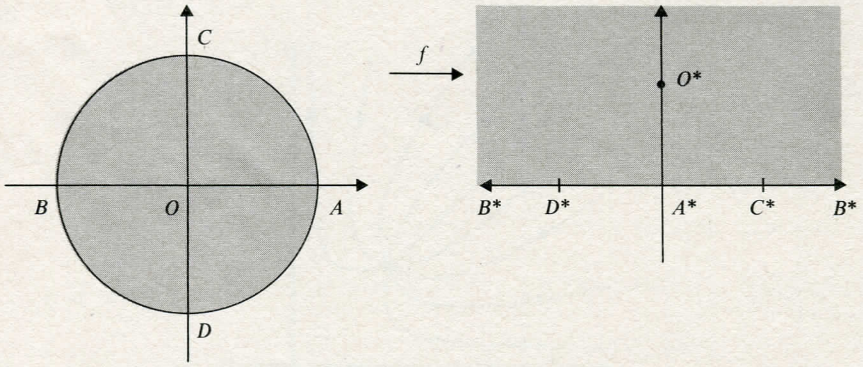
1.



$R = \{z : 0 < \operatorname{Arg} z < \pi/n\}$
$f : z \longrightarrow z^n$

$S = \{w : \operatorname{Im} w > 0\}$
$g : w \longrightarrow w^{1/n}$(principal branch)

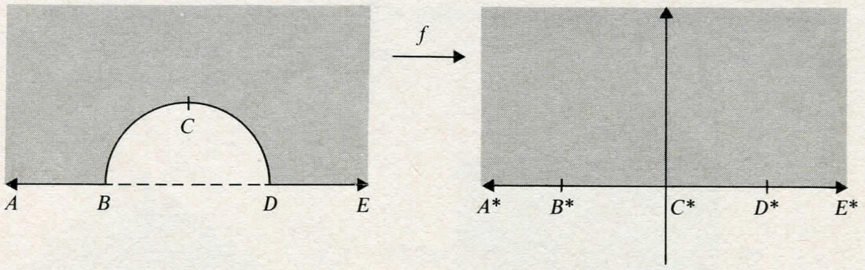
2.



$R = \{z : |z| < 1\}$
$f : z \longrightarrow \frac{1}{i} \cdot \frac{z - 1}{z + 1}$

$S = \{w : \operatorname{Im} w > 0\}$
$g : w \longrightarrow \frac{1 + iw}{1 - iw}$

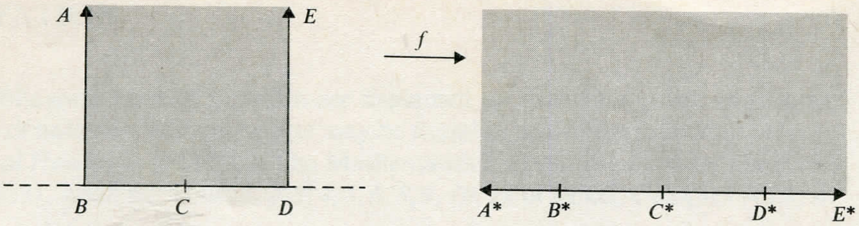
3.



$R = \{z : \operatorname{Im} z > 0, |z| > 1\}$
$f : z \longrightarrow z + \frac{1}{z}$

$S = \{w : \operatorname{Im} w > 0\}$
$g : w \longrightarrow \frac{w + \sqrt{w^2 - 4}}{2}$

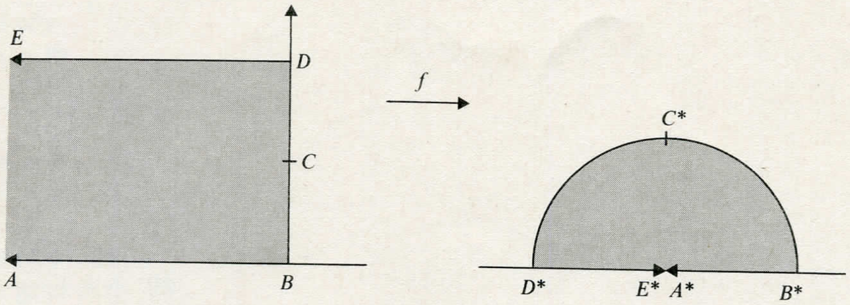
4.



$R = \{z : \operatorname{Im} z > 0, |\operatorname{Re} z| < \pi/2\}$
$f : z \longrightarrow \sin z$

$S = \{w : \operatorname{Im} w > 0\}$
$g : w \longrightarrow \sin^{-1} w$
(appropriate branch)

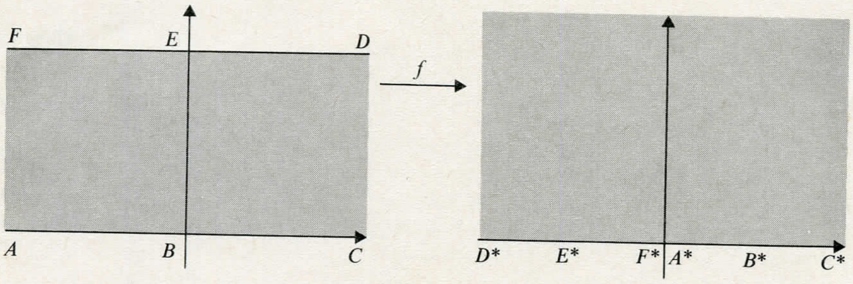
5.



$R = \{z : \operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi\}$
$f : z \longrightarrow \exp z$

$S = \{w : |w| < 1, \operatorname{Im} w > 0\}$
$g : w \longrightarrow \operatorname{Log} w$

6.



$R = \{z : 0 < \operatorname{Im} z < \pi\}$
$f : z \longrightarrow \exp z$

$S = \{w : \operatorname{Im} w > 0\}$
$g : w \longrightarrow \operatorname{Log} w$

NOTATION

The following items of notation are explained on the pages given. Most other items of notation used in the text may be found in one of the following: *Mathematical Handbook* for M100, The Mathematics Foundation Course; *Handbook* for M231, Analysis; *Units 1, 2, 3, 4, 5, 6, 8, 9, 10, 11* of M332, Complex Analysis.

$\frac{\partial F}{\partial x}(a, b), \frac{\partial F}{\partial x}$	7
$\frac{\partial}{\partial x}(G)$	7
$\frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial x \partial y}$	7
$\nabla_\theta F(\alpha)$	10
$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}(F(x))$	13
$\mathbf{a} \bullet \mathbf{b}, \mathbf{a} \wedge \mathbf{b}$	25
$a \bullet b, a \wedge b$	26
$\mathbf{F}(z)$	28
$\nabla\phi$, grad ϕ	28
$\nabla\phi(z)$, grad $\phi(z)$	28
$\mathbf{t}$	29
$\mathbf{n}$	29
div F, [div F](x, y)	31
curl F, [curl F](x, y)	31
$\int_\Gamma f(z)\,dz$	34
$\int_\Gamma \mathbf{F}(z) \bullet dz$	34
$\int_\Gamma \mathbf{F}(z) \wedge dz$	34
$\frac{\partial \phi}{\partial n}, \frac{\partial \phi}{\partial n}(z)$	39
$\lim_{\zeta \to z \text{ in } R} U(\zeta)$	42

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COMPLEX ANALYSIS

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3 Differentiation
4 Integration
5 Cauchy's Theorem I
6 Taylor Series
7 NO TEXT
8 Singularities
9 Cauchy's Theorem II
10 The Calculus of Residues
11 Analytic Functions
12 NO TEXT
13 Properties of Analytic Functions
14 Laplace Transforms
15 Number Theory
16 Boundary Value Problems

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